Math 80220 Algebraic Number Theory
Problem Set 5

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due Wednesday, April 2

1. Consider the functions $\mathcal{B}(x) = \dfrac{x}{e^x - 1}$ and $\mathcal{B}(x, z) = \dfrac{xe^{xz}}{e^x - 1}$.

   (a) Show that the Taylor expansion of $\mathcal{B}(x)$ around $x = 0$ is of the form

   $$\frac{x}{e^x - 1} = \sum_{n \geq 0} B_n \frac{x^n}{n!}$$

   where $B_n \in \mathbb{Q}$ are such that $B_{2k+1} = 0$ if $k \geq 1$. The coefficients B_n are known as the **Bernoulli numbers**. [Hint: Show that $\mathcal{B}(x) + x/2$ is even.]

   (b) Show that "$B^n = (B + 1)^n$" is satisfied by the Bernoulli numbers in the sense that

   $$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k$$

   when $n \neq 1$. [Hint: Compute the power series product $\mathcal{B}(x) \exp(x)$.]

   (c) Show that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$ and $B_4 = -\frac{1}{30}$. [Hint: work $\pmod{x^2}$ and then use the previous part.]

   (d) Show that

   $$\mathcal{B}(x, z) = \sum_{n \geq 0} B_n(z) \frac{x^n}{n!}$$

   where $B_n(z) = \displaystyle\sum_{k=0}^{n} \binom{n}{k} B_{n-k} z^k$ are polynomials in z of degree n (called the Bernoulli polynomials).

   (e) With B_n and $B_n(z)$ as above, show that for $n, m \geq 1$ (with the convention $0^0 = 1$) one has

   $$\begin{aligned}\sum_{k=0}^{n-1} k^{m-1} &= \frac{B_m(n) - B_m}{m} \\ &= \frac{1}{m} \sum_{k=0}^{m-1} \binom{m}{k} B_k n^{m-k}\end{aligned}$$

   [Hint: Take $\sum_{m \geq 1} (\cdot) \frac{x^m}{m!}$ of the above expressions and compare generating functions.]

2. (Optional, but you need the results for Problem 3) Consider the Euler Γ-function $\Gamma(x) = \displaystyle\int_0^\infty t^{x-1} e^{-x} dx$ and the Euler B-function $B(x, y) = \displaystyle\int_0^1 t^{x-1} (1 - t)^{y-1} dt$.

(a) Show that $\Gamma(x+1) = x\Gamma(x)$ and conclude that $\Gamma(n) = (n-1)!$ for $n \geq 1$.

(b) Show that $\Gamma(x)\Gamma(y) = \Gamma(x+y)B(x+y)$.

(c) You may assume that $B(x,y)B(x+y,1-x) = \dfrac{\pi}{x\sin(\pi y)}$. Show that $\Gamma(1-z)\Gamma(z) = B(1-z,z) = \dfrac{\pi}{\sin(\pi z)}$ and conclude that $\Gamma(1-n) = \dfrac{(-1)^{n+1}\pi}{(n-1)!}$.

(d) Show that

$$B(x,y) = 2\int_0^{\pi/2} \cos^{2x-1}(\theta)\sin^{2y-1}(\theta)d\theta$$

[Hint: Change variables $x = \cos^2(\theta)$.]

(e) Show that $B(z,z) = 2^{2-2z}\int_0^1 (1-x^2)^{z-1}dx$. [Hint: Change variables $t = (1+x)/2$.]

(f) Show that $\int_0^1 (1-x^2)^{z-1}dx = B(1/2,z)$ and conclude that $B(z,z) = 2^{1-2z}B(1/2,z)$. [Hint: change variables $x = \cos(\theta)$ and use part 2d.]

(g) Show that $\Gamma(1/2) = \sqrt{\pi}$ and conclude that $\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$. [Hint: Compute $B(1/2,1/2)$ and then use 2f.]

(h) Deduce that $\Gamma(1/2-n) = \dfrac{(-4)^n n!\sqrt{\pi}}{(2n)!}$.

3. Recall that

$$\sin(z) = z\prod_{n\geq 1}\left(1 - \frac{z^2}{n^2\pi^2}\right)$$

(a) Show that

$$z\,\mathrm{cotan}(z) = 1 + 2\sum_{n=1}^{\infty}\frac{z^2}{z^2 - n^2\pi^2}$$

[Hint: Take logarithmic derivative of $\sin(z)$.]

(b) Show that

$$z\,\mathrm{cotan}(z) = 1 + \sum_{n\geq 2}(2i)^n B_n \frac{z^n}{n!}$$

[Hint: Plug in $x = 2iz$ in $\mathcal{B}(x)$ and recall that $e^{iz} = \cos(z) + i\sin(z)$.]

(c) Show that

$$\frac{z^2}{z^2 - n^2\pi^2} = -\sum_{k\geq 1}\frac{z^{2k}}{n^{2k}\pi^{2k}}$$

and conclude that

$$z\,\mathrm{cotan}(z) = 1 - 2\sum_{k\geq 1}\frac{z^{2k}\zeta(2k)}{\pi^{2k}}$$

(d) Deduce that for $n \geq 1$,

$$\zeta(2n) = \frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!}$$

(e) Show that for $n \geq 1$

$$\zeta(1-2n) = -\frac{B_{2n}}{2n}$$

and thus make sense of the famous expression

$$1 + 2 + 3 + \cdots = -\frac{1}{12}$$

[Hint: Use the functional equation and the previous exercise (even if you didn't do the previous exercise).]

4. (a) Show that

$$\pi \operatorname{cotan}(\pi z) = \frac{1}{z} + \sum_{n \geq 1} \left(\frac{1}{z+n} + \frac{1}{z-n} \right)$$

[Hint: Use the previous exercise.]

(b) Writing $q = \exp(2\pi i z)$ show that

$$\pi \operatorname{cotan}(\pi z) = \pi i - 2\pi i \sum_{n=0}^{\infty} q^n$$

[Hint: What are $\sin(\pi z)$ and $\cos(\pi z)$ in terms of q?]

(c) Deduce that for $k \geq 2$

$$\sum_{n \in \mathbb{Z}} \frac{1}{(n+z)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^n$$

[Hint: Differentiate the two expressions for $\pi \operatorname{cotan}(\pi z)$.]

(d) For $k \geq 1$ consider the expression $G_{2k}(z) = \displaystyle\sum_{(m,n) \neq (0,0), m,n \in \mathbb{Z}} \frac{1}{(mz+n)^{2k}}$. Show that

$$G_{2k}(z) = 2\zeta(2k) + 2 \sum_{m \geq 1, n \in \mathbb{Z}} \frac{1}{(mz+n)^{2k}}$$

and conclude that

$$G_{2k}(z) = 2\zeta(2k) + \frac{2(-2\pi i)^{2k}}{(2k-1)!} \sum_{n \geq 1} \sigma_{2k-1}(n) q^n$$

where $\sigma_r(n) = \sum_{d|n} d^r$.

(e) Show that $E_{2k} = \dfrac{(2k-1)!}{2(-2\pi i)^{2k}} G_{2k}$ satisfies

$$E_{2k} = \frac{\zeta(1-2k)}{2} + \sum_{n \geq 1} \sigma_{2k-1}(n) q^n$$

and so is in $\mathbb{Q}[\![q]\!]$.

(f) Show that $E_{12} \equiv \sum_{n \geq 1} \sigma_{11}(n) q^n \pmod{691}$ and thus lies in $q\mathbb{F}_{691}[\![q]\!]$. (You may look up the Bernoulli number B_{12}.)

(g) (Optional) Show, directly from the definition, that if $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = 1$ then

$$G_{2k}\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} G_{2k}(z)$$

Remark 1. Part 4g shows that G_{2k} and E_{2k} are *modular forms*. Part 4f implies one of the Ramanujan identities, that if $\tau(n)$ is the coefficient of q^n in $\Delta = q \prod_{n \geq 1} (1-q^n)^{24}$ then $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ (one shows that $E_{12} \equiv \Delta \pmod{691}$) because both power series start with $q \pmod{691}$).