Introduction to Algebraic Number Theory
Lecture 6

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Today: traces and norms, discriminants and integral bases. Textbook here is `http://wstein.org/books/ant/ant.pdf`

4 Dedekind domains

(4.4)

Theorem 1. *If $K/\mathbb{Q}$ is a number field then $\mathcal{O}_K$ is a Dedekind domain.*

Proof. Done in class. See textbook Proposition 3.1.5. □

(4.5) Fractional ideals.

Definition 2. Let R be a ring and I, J two ideals. Say that $I \mid J$ if $J \subset I$.

Lemma 3. *Let R be a noetherian ring. Then every ideal I of R divides a product of prime ideals.*

Proof. Done in class. See textbook Lemma 3.1.10 which really only uses the noetherian property. □

Theorem 4. *If R is a Dedekind domain then every fractional ideal is invertible, i.e., the set of fractional ideals is a group.*

Proof. I did this in three steps.

Step 1: We do this for $I = \mathfrak{p}$ a prime ideal of R.

Step 2: Do this for I an ideal of R.

Step 3: Do this for I a fractional ideal of R. This last step is easy, since there exists $\alpha \in R$ nonzero such that αI is an ideal. Then αI is invertible and $I^{-1} = \alpha(\alpha I)^{-1}$.

Steps 1 and 2 are done in the textbook, proof of Theorem 3.1.8 on page 45 where it's phrased only for rings of integers of number fields but the proof is identical in the case of Dedekind domains. □

A feature of the proof of the above theorem: If $\prod \mathfrak{p}_i \subset \mathfrak{p}_p$ where $\mathfrak{p}$ and $\mathfrak{p}_i$ are prime ideals then $\mathfrak{p} = \mathfrak{p}_i$ for some i.