Introduction to Algebraic Number Theory
Lecture 19

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8 Units (continued)

(3.1) Proof of the Dirichlet unit theorem.

Lemma 1. *1. There exists a constant λ (in fact $\lambda = \log\left(2^s\pi^{-s}\sqrt{|\operatorname{disc}(K)|}\right)$) such that for any index k between 1 and $r+s$ and any $\alpha = (a_1, \dots, a_{r+s}) \in \log\iota(\mathcal{O}_K - 0)$ there exists $\beta = (b_1, \dots, b_{r+s}) \in \log\iota(\mathcal{O}_K^\times - 0)$ with $\sum\beta < \lambda$ and $b_i < a_i$ for all $i \neq k$.*

2. For any index k there exists $\alpha = (u_1, \dots, u_{r+s}) \in \log\iota\mathcal{O}_K^\times$ such that $u_i < 0$ when $i \neq k$.

Proof. Part one follows from a geometry of numbers type argument. Here's a sketch: choose c_i such that $c_i < \exp(a_i)$ for $i \neq k$ and choose c_k such that $\prod c_i = \exp(\lambda)$. Then finding β as desired is equivalent to finding $x = (x_1, \dots, x_n) \in \iota(\mathcal{O}_K - 0)$ such that $|x_i| < c_i$ for $i \leq r$ and $x_{r+2i-1}^2 + x_{r+2i}^2 < c_{r+i}$ for $i > 0$. The geometry of numbers requires only that the volume of this region be $> 2^n \operatorname{vol}(\iota(\mathcal{O}_K))$ and the volume can be shown to depend only on λ. For example, if $K = \mathbb{Q}(\sqrt{m})$, $m > 0$ then $\mathbb{R}^n = \mathbb{R}^{r+s} = \mathbb{R}^2$ and the region $|x_i| \leq c_i$ with $c_1 \exp(a_1)$ and $c_1c_2 = \exp(\lambda)$ has area $4c_1c_2 = 4\exp(\lambda)$.

Part two: part one allows us to construct a sequence $\alpha_m = (a_{m,1}, \dots, a_{m,r+s}) \ni \log\iota(\mathcal{O}_K - 0)$ with $(a_{m,i})_m$ decreasing for $i \neq k$ and $\sum\alpha_m < \lambda$. Consider the $\sum$ map $\sum : \log\iota(\mathcal{O}_K - 0) \to \mathbb{R}$ taking $\log\iota(\alpha)$ to $\log|N_{K/\mathbb{Q}}(\alpha)|$. If $B > 0$ then $\sum\log\iota(\alpha) \leq B$ implies $|N_{K/\mathbb{Q}}(\alpha)| \leq \exp(B)$ and so $N_{K/\mathbb{Q}}(\alpha)$ takes finitely many integral values (between $-\exp(B)$ and $\exp(B)$). Using the observation with $B = \lambda$ it follows that if $\alpha_m = \log\iota(u_m)$ for $u_m \in \mathcal{O}_K - 0$ then $|N_{K/\mathbb{Q}}(u_m)| \leq \exp(\lambda)$ for all m. By the pigeonhole principle there must exist different indices $m \neq m'$ such that $|N_{K/\mathbb{Q}}(u_m)| = |N_{K/\mathbb{Q}}(u_{m'})|$ which implies that $u_m = uu_{m'}$ for some $u \in \mathcal{O}_K^\times$. In other words $\alpha_m = \log\iota(u) + \alpha_{m'}$ for $m \neq m'$ for some unit $u \in \mathcal{O}_K^\times$ and the condition on u follows from the fact that the coordinates of α_m are decreasing for $i \neq k$. □

Proof of the Dirichlet Unit Theorem. It suffices to show that $\mathcal{O}_K^\times$ has rank at least $r+s-1$. The previous lemma guarantees the existence of units u_k such that $\log\iota(u_k)$ have negative coordinates except in index k. Since $\sum\log\iota(u_k) = 0$ it follows that the k-th coordinate of $\log\iota(u_k)$ must be > 0.

Consider the matrix $(u_{i,j})$ where $\log\iota(u_i) = (u_{i,1}, \dots, u_{i,r+s})$. To show that $\operatorname{rank}\mathcal{O}_K^\times = r+s-1$ it suffices to show that $r+s-1$ of the $\log\iota(u_k)$ are linearly independent, i.e., the rank of this matrix is $\geq r+s-1$.

Suppose the rank is $< r+s-1$ in which case we may assume that there exist $t_1, t_2, \dots, t_{r+s-s}$ such that

$\sum_{j=1}^{r+s-1} t_j u_{i,j} = 0$ for all i. We may assume that the largest coefficient $t_k > 0$. Then

$$
\begin{aligned}
0 &= \sum_{j=1}^{r+s-1} t_j u_{k,j} \\
&= t_k u_{k,k} + \sum_{j \neq k, 1 \leq j \leq r+s-1} t_j u_{k,j} \\
&\geq t_k u_{k,k} + \sum_{j \neq k, 1 \leq j \leq r+s-1} t_k u_{k,j} \\
&= t_k \sum_{j=1}^{r+s-1} u_{k,j} \\
&= -t_k u_{k,r+s}
\end{aligned}
$$

since $u_{k,j} < 0$ when $j \neq k$ and $\sum_{j=1}^{r+s} u_{k,j} = 0$ for all k. This of course is not possible since $t_k > 0$ and $u_{k,r+s} < 0$ as $k < r + s$. □