Introduction to Algebraic Number Theory
Lecture 23

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10 ζ-functions and L-functions

(10.3) Functional equation (continued).

Corollary 1 (A basic version of Birch and Swinnerton-Dyer). *The function ζ_K has a zero of order r_1+r_2-1 at $s=0$ and*

$$\frac{\zeta_K^{(r_1+r_2-1)}(0)}{(r_1+r_2-1)!}=\frac{h_K R_K}{w}$$

Here the order of vanishing r_1+r_2-1 is the rank of the finitely generated abelian group $\mathcal{O}_K^\times=\mathrm{Res}_{K/\mathbb{Q}}\,\mathbb{G}_m(\mathbb{Z})$.

Proof. Using the previous lemma on the Γ-function we get $\Gamma_\mathbb{R}(s)=\frac{2\pi}{s}\Gamma_\mathbb{R}(s+2)$ and $\Gamma_\mathbb{C}(s)=\frac{2\pi}{s}\Gamma_\mathbb{C}(s+1)$ have simple poles at $s=0$. These formulae transform the functional equation into

$$d_K^{s/2}\frac{(2\pi)^{r_1}}{s^{r_1}}\Gamma_\mathbb{R}\left(s+2\right)^{r_1}\frac{(2\pi)^{r_2}}{s^{r_2}}\Gamma_\mathbb{C}(s+1)^{r_2}\zeta_K(s)=d_K^{(1-s)/2}\Gamma_\mathbb{R}\left(1-s\right)^{r_1}\Gamma_\mathbb{C}(1-s)^{r_2}\zeta_K(1-s)$$

Recall that ζ_K has a simple pole at $s=1$ and so $\zeta_K(1-s)=\dfrac{f(s)}{s}$ where $f(0)=\dfrac{2^{r_1}(2\pi)^{r_2}h_K R_K}{w\sqrt{|\,\mathrm{disc}(K)|}}$. Therefore

$$\zeta_K(s)=\frac{s^{r_1+r_2-1}f(s)d_K^{(1-2s)/2}\Gamma_\mathbb{R}\left(1-s\right)^{r_1}\Gamma_\mathbb{C}(1-s)^{r_2}}{2^{r_1+r_2}\pi^{r_1+r_2}\Gamma_\mathbb{R}\left(s+2\right)^{r_1}\Gamma_\mathbb{C}(s+1)^{r_2}}$$

In this expression the only factor vanishing at $s=0$ is $s^{r_1+r_2-1}$ and so the order of vanishing is as desired. Taking derivatives at 0 we get

$$\begin{aligned}\frac{\zeta_K^{(r_1+r_2-1)}(0)}{(r_1+r_2-1)!}&=\frac{f(0)\sqrt{d_K}\Gamma_\mathbb{R}(1)^{r_1}\Gamma_\mathbb{C}(1)^{r_2}}{2^{r_1+r_2}\pi^{r_1+r_2}\Gamma_\mathbb{R}(2)^{r_1}\Gamma_\mathbb{C}(1)^{r_2}}\\&=\frac{f(0)\sqrt{d_K}\pi^{-r_2}}{2^{r_1+r_2}\pi^{r_1+r_2}\pi^{-r_1}\pi^{-r_2}}\\&=\frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{2^{r_1+r_2}\pi^{r_2}w}\\&=\frac{h_K R_K}{w}\end{aligned}$$

using $\Gamma_\mathbb{R}(1)=\pi^{-1/2}\Gamma(1/2)=1$, $\Gamma_\mathbb{C}(1)=2(2\pi)^{-1}=\pi^{-1}$ and $\Gamma_\mathbb{R}(2)=\pi^{-1}$.

□

(10.4) Characters.

Definition 2. Let G be a finite abelian group. A **character** of G is a group homomorphism $\chi:G\to\mathbb{C}^\times$.

Proposition 3. *The set $\widehat{G}$ of all characters of G is a finite abelian group.*

1. *If G and H are finite abelian groups then $\widehat{G \times H} \cong \widehat{G} \times \widehat{H}$.*
2. *There is a non-canonical isomorphism $G \approx \widehat{G}$.*
3. *There is a canonical isomorphism $\widehat{\widehat{G}} \cong G$ given by $g \mapsto (\chi \mapsto \chi(g))$.*

Proof. Part one: If $\chi \in \widehat{G \times H}$ let $\chi_1(g) = \chi(g, 1)$ and $\chi_2(h) = \chi(1, h)$. Then $\chi(g, h) = \chi_1(g)\chi_2(h)$ and we get a map $\widehat{G \times H} \to \widehat{G} \times \widehat{H}$ given by $\chi \mapsto \chi_1 \times \chi_2$. This is clearly an isomorphism.

Part two: If $G = \mathbb{Z}/n\mathbb{Z}$ then every $\chi \in \widehat{G}$ is uniquely defined by $\chi(1) \in \mu_n$ and so $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mu_n$. But $\mu_n \approx \mathbb{Z}/n\mathbb{Z}$ identifying ζ_n^k with k for a choice of primitive n-th root ζ_n. The result now follows from this and part one.

Part three: The given map is a canonical homomorphism. Suppose it is not injective. Then there exists $g \in G$ such that $\chi(g) = 1$ for every $\chi \in \widehat{G}$. But there is a nontrivial character of $\langle g \rangle$ sending g to a primitive root of 1 of order equal to the order of g. This defines a character $G \twoheadrightarrow \langle g \rangle \to \mathbb{C}^\times$ which is not trivial on g. Finally, we have an injective homomorphism between finite sets of the same size (from part two) and so it is an isomorphism. □

(10.5) L-functions.

Definition 4. A character mod N is a character of the group $(\mathbb{Z}/N\mathbb{Z})^\times$, i.e., a homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. The L-function of χ is the Dirichlet series

$$L(\chi, s) = \sum_{(n,N)=1} \frac{\chi(n)}{n^s}$$

which converges to a holomorphic function on $\operatorname{Re} s > 1$.

Proposition 5. *If χ is a character mod N which is not the trivial character then*

1. $\sum_{k \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(k) = 0$
2. *$L(\chi, s)$ is analytic when $\operatorname{Re} s > 0$.*

Proof. If G is any finite abelian group and $\chi : G \to \mathbb{C}^\times$ is any nontrivial homomorphism then $\sum_{g \in G} \chi(g) = 0$. Indeed, $\chi \neq 1$ implies that $\chi(h) \neq 1$ for some $h \in G$. Then

$$\begin{aligned} \sum_{g \in G} \chi(g) &= \sum_{gh \in G} \chi(gh) \\ &= \chi(h) \sum_{g \in G} \chi(g) \end{aligned}$$

and so the sum must vanish.

Look at the partial sums $A_t = \sum_{n=1}^{t} \chi(t)$. The sum over representatives of $\mathbb{Z}/N\mathbb{Z}$ is 0 and so $|A_t| \leq \sum |\chi(k)| = \varphi(N)$ is bounded. Thus $L(\chi, s)$ is holomorphic for $\operatorname{Re} s > 0$. □