Introduction to Algebraic Number Theory
Lecture 24

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10 ζ-functions and L-functions

(10.6) Euler products.

Proposition 1. *Have*

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{||\mathfrak{p}||^s}\right)^{-1}$$

$$L(\chi, s) = \prod_{p \nmid N} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

where p is a prime and $\mathfrak{p}$ is a prime ideal.

Proof. Follows from unique factorization in Dedekind domains and the fact that χ is a homomorphism. □

Proposition 2. *Suppose G is a finite abelian group and $N > 1$ an integer.*

1. *If $\chi \in \widehat{G}$ then*

$$\prod_{g \in G} (X - \chi(g)) = (X^a - 1)^b$$

   where $a = |\operatorname{Im}\chi|$ is the order of χ in $\widehat{G}$ and $b = |\ker\chi| = |G|/a$.

2. *If $p \in (\mathbb{Z}/N\mathbb{Z})^\times$ has order exactly r then*

$$\prod_{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} (X - \chi(p)) = (X^r - 1)^{\varphi(N)/r}$$

Proof. Part one: Since χ is a homomorphism, $\operatorname{Im}\chi \subset \mathbb{C}^\times$ is a subgroup and thus isomorphic to μ_a for $a = |\operatorname{Im}\chi|$. Now

$$\begin{aligned}
\prod_{g \in G}(X - \chi(g)) &= \prod_{g \in \ker\chi} \prod_{h \in G/\ker\chi} (X - \chi(gh)) \\
&= \prod_{h \in G/\ker\chi} (X - \chi(h))^b \\
&= \prod_{\zeta \in \operatorname{Im}\chi \cong \mu_a} (X - \zeta)^b \\
&= (X^a - 1)^b
\end{aligned}$$

Finally, the statements about a being the order of χ in $\widehat{G}$ and $ab = |G|$ are immediate from the fact that $\operatorname{Im}\chi = \mu_a$ and the first isomorphism theorem.

Part two: Let $G = (\mathbb{Z}/N\mathbb{Z})^\times$ and $\psi_p \in \widehat{\widehat{G}}$ given by $\psi_p(\chi) = \psi(p)$ for any $\psi \in \widehat{G}$. Then $\prod_{\chi\in\widehat{G}} (X - \chi(p)) = \prod_{\chi\in\widehat{G}} (X - \psi_p(\chi))$. We will apply part one to the group $\widehat{G}$ and the element ψ_p and deduce that this product is $(X^a - 1)^b$ where $a = |\operatorname{Im}\psi_p|$ is the order of ψ_p and $b = |\ker\psi_p| = \varphi(N)/a$. But the order of ψ_p in $\widehat{\widehat{G}}$ is the same as the order r of p in G. □

Theorem 3. *Suppose $K = \mathbb{Q}(\zeta_N)$ for $N > 1$. Then*

$$\zeta_K(s) = \prod_{\mathfrak{p}|N}\left(1 - \frac{1}{||\mathfrak{p}||^s}\right)^{-1} \prod_{\chi\in\widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} L(\chi, s)$$

where

$$\zeta(s) = \prod_{p\nmid N}\left(1 - \frac{1}{p^s}\right)^{-1} L(1, s)$$

for the trivial mod N *character.*

Proof. We only need to show that

$$\prod_{\mathfrak{p}\nmid N}\left(1 - \frac{1}{||\mathfrak{p}||^s}\right)^{-1} = \prod_{\chi\in\widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} L(\chi, s)$$

which is equivalent to

$$\prod_{\mathfrak{p}\nmid N}\left(1 - \frac{1}{||\mathfrak{p}||^s}\right)^{-1} = \prod_{\chi\in\widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} \prod_{p\nmid N}\left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

For this it suffices to show that

$$\prod_{\chi\in\widehat{(\mathbb{Z}/N\mathbb{Z})^\times}}\left(1 - \frac{\chi(p)}{p^s}\right) = \prod_{\mathfrak{p}|p}\left(1 - \frac{1}{||\mathfrak{p}||^s}\right)$$

Let e and f be the ramification and inertia indices of $\mathfrak{p}$ (independent of $\mathfrak{p} \mid p$ since $K/\mathbb{Q}$ is Galois). Then the RHS is $\left(1 - \frac{1}{p^{fs}}\right)^e$ and so it suffices to show that

$$\prod_{\chi\in\widehat{(\mathbb{Z}/N\mathbb{Z})^\times}} (p^s - \chi(p)) = (p^{fs} - 1)^e$$

By a previous proposition the LHS is $(p^{as} - 1)^b$ where a is the order of p in $(\mathbb{Z}/N\mathbb{Z})^\times$ and $b = \varphi(N)/a$. Thus is suffices to show that $a = f$.

But f is the degree of any irreducible factor of the cyclotomic polynomial Φ_N in $\mathbb{F}_p[X]$ (since $\mathcal{O}_{\mathbb{Q}(\zeta_N)} = \mathbb{Z}[\zeta_N]$ so there is no restriction on p). Such an irreducible factor has as roots primitive N-th roots of 1 which would then be defined over $\mathbb{F}_{p^f}$ (and no smaller subfield) and the result follows as on the homework from the fact that $\mathbb{F}_{p^f}^\times \cong \mathbb{Z}/(p^f - 1)\mathbb{Z}$ and so $N \mid p^f - 1$ (but not so for smaller exponents) which is equivalent to f being the order a of p. □

Corollary 4. *If χ is a nontrivial character* mod N *then $L(\chi, 1) \neq 1$.*

Proof. From the previous theorem we get

$$(s-1)\zeta_K(s) = (s-1)\zeta(s)\prod_{p\nmid N}\left(1-\frac{1}{p^s}\right)\prod_{\mathfrak{p}|N}\left(1-\frac{1}{||\mathfrak{p}||^s}\right)^{-1}\prod_{\chi\neq 1}L(\chi,s)$$

Taking $\lim_{s\to 1}$ we get that $\prod_{\chi\neq 1}L(\chi,1)$ is nonzero. □

(10.7) Density.

Lemma 5. *Let $K/\mathbb{Q}$ be a number field. As $s\to 1^+$ we have the estimate*

$$\sum_{\mathfrak{p}}\frac{1}{||\mathfrak{p}||^s} = \log\zeta_K(s) + O(1) = -\log(s-1) + O(1)$$

Proof. We have

$$\begin{aligned}
\log\zeta_K(s) &= \log\prod_{\mathfrak{p}}\left(1-\frac{1}{||\mathfrak{p}||^s}\right)^{-1}\\
&= -\sum_{\mathfrak{p}}\log\left(1-\frac{1}{||\mathfrak{p}||^s}\right)\\
&= \sum_{\mathfrak{p}}\sum_{n\geq 1}\frac{1}{n||\mathfrak{p}||^{ns}}\\
&= \sum_{\mathfrak{p}}\frac{1}{||\mathfrak{p}||^s} + \sum_{\mathfrak{p}}\sum_{n\geq 2}\frac{1}{n||\mathfrak{p}||^{ns}}
\end{aligned}$$

and so

$$\begin{aligned}
\left|\log\zeta_K(s) - \sum_{\mathfrak{p}}\frac{1}{||\mathfrak{p}||^s}\right| &\leq \sum_{\mathfrak{p}}\sum_{n\geq 2}\frac{1}{n||\mathfrak{p}||^{ns}}\\
&< \sum_{\mathfrak{p}}\frac{1}{||\mathfrak{p}||^s(||\mathfrak{p}||^s-1)}\\
&< \sum_{\mathfrak{p}}\frac{2}{||\mathfrak{p}||^{2s}}\\
&< 2\zeta_K(2s)
\end{aligned}$$

since $||\mathfrak{p}||^s > 2$ and so $||\mathfrak{p}||^s - 1 > ||\mathfrak{p}||^s/2$.

The first estimate then follows from the fact that $\zeta_K(s)$ is holomorphic around $s=2$. The second estimate follows from the fact that $\zeta_K(s)$ has a simple pole at $s=1$. □