Introduction to Algebraic Number Theory
Lecture 29

Andrei Jorza

12 Cyclotomic units

This is from Washington, Introduction to Cyclotomic Fields.

(12.1) Let V_n be the group generated by $\pm\zeta_n$ and $1-\zeta_n^a$ for all $1<a<n$.

Remark 1. $V_n \subset \mathbb{Z}[\zeta_n]^\times$.

Definition 1. If $K/\mathbb{Q}$ is a number field with abelian Galois group over $\mathbb{Q}$ then the *cyclotomic units* of K are $V_n \cap \mathcal{O}_K^\times$ for n such that $K \subset \mathbb{Q}(\zeta_n)$. Denote the group of cyclotomic units C_K.

Let $p>2$ be a prime and $m \geq 1$. Consider the number fields $K=\mathbb{Q}(\zeta_{p^m})$ totally complex of degree $p^{m-1}(p-1)/2$ over $\mathbb{Q}$ and $K^+=\mathbb{Q}(\zeta_{p^m})^+=\mathbb{Q}(\zeta_{p^m}+\zeta_{p^m}^{-1})$ totally real of degree $p^{m-1}(p-1)/2$ over $\mathbb{Q}$. Note that K/K^+ is quadratic generated by an imaginary quadratic number. Then

$$\operatorname{rank}\mathcal{O}_K^\times = \operatorname{rank}\mathcal{O}_{K^+}^\times = p^{m-1}(p-1)/2-1$$

Lemma 2. *For $1<a<p^m/2$ coprime to p let*

$$\xi_a = \zeta_{p^m}^{(1-a)/2}\frac{1-\zeta_{p^m}^a}{1-\zeta_{p^m}}$$

1. *$\xi_a \in \mathcal{O}_{K^+}^\times$.*
2. *The group C_K of cyclotomic units of K are generated by $\pm\zeta_{p^m}$ and the units ξ_a for $1<a<p^m/2$ coprime to p.*
3. *The group C_{K^+} of cyclotomic units of K^+ are generated by -1 and the units ξ_a for $1<a<p^m/2$ coprime to p.*

Proof. Part one: First, write $\zeta=\zeta_{p^m}$ in which case

$$\begin{aligned}\zeta^{1/2} &= e^{\pi i/p^m}\\ &= e^{2\pi i+\pi i/p^m}\\ &= -e^{2\pi i(p^m+1)/(2p^m)}\\ &= -\zeta^{(p^m+1)/2}\in K\end{aligned}$$

Next, it is clear that ξ_a is real and so $\xi_a \in K^+$ and it is invertible because $1-\zeta \mid 1-\zeta^a$ and since $(a,p)=1$ it follows that $ab=pk+1$ for some k in which case $1-\zeta^a \mid 1-\zeta = 1-(\zeta^a)^b$.

Part two: C_K is generated by $\pm\zeta$ and $1-\zeta^a$ for some a. Write $a=bp^k$ with $p \nmid p$. Note that

$$1-X^{p^k} = \prod_{j=0}^{p^k-1}(1-\zeta_{p^k}^jX) = \prod_j (1-\zeta^{jp^{m-k}}X)$$

and so

$$1-\zeta^a = \prod_j (1-\zeta^{b+jp^{m-k}})$$

If $k < m$, i.e., if $p^m \nmid a$ then the RHS consists of factors of the type $1-\zeta^a$ with $(a,p)=1$ and so C_K is generated by $\pm\zeta$ and $1-\zeta^a$ for $(a,p)=1$. Moreover, $1-\zeta^{-a} = -\zeta^{-a}(1-\zeta^a)$ so C_K is generated by $\pm\zeta$ and $1-\zeta^a$ for $1 \le a < p^m/2$ coprime to p.

Suppose $\xi \in C_K$. Then

$$\xi = \pm\zeta^k \prod_{1<a<p^m/2,(a,p)=1} (1-\zeta^a)^{c_a}$$

and on the level of ideals $(1) = (\xi) = \prod(1-\zeta^a)^{c_a}$. But $(1-\zeta^a) = (1-\zeta)$ and so $\sum c_a = 0$. Thus

$$\xi = \pm\zeta^{k+\sum c_a(a-1)/2} \prod_{1<a<p^m/2,(a,p)=1} \xi_a^{c_a}$$

Part three: we only require that ξ be real in which case $k=0$ in the above formula.

□

(12.2) Cyclotomic units of K^+.

Lemma 3. *Let G be a finite abelian group and $f : G \to \mathbb{C}$ be any function. Then*

$$\det(f(\sigma\tau^{-1}) - f(\sigma))_{\sigma,\tau\neq 1} = \prod_{\chi\in\widehat{G},\chi\neq 1} \sum_{\sigma\in G} \chi(\sigma)f(\sigma)$$

Proof. Do an example with $G = \mathbb{Z}/3\mathbb{Z}$. □