Introduction to Algebraic Number Theory
Lecture 30

Andrei Jorza

12 Cyclotomic units

(12.3) Cyclotomic units of K^+ (continued).

Theorem 1. *The group C_{K^+} of cyclotomic units of K^+ is a finite index subgroup of $\mathcal{O}_{K^+}^\times$ and*

$$[\mathcal{O}_{K^+}^\times : C_{K^+}] = h_{K^+}$$

is the class number.

Proof. Recall the map $\log \circ \iota : K^+ \to \mathbb{R}^{p^{m-1}(p-1)/2}$ with kernel ± 1, the roots of unity in K^+. Also recall that $\log \iota \mathcal{O}_{K^+}^\times \subset \Delta = \ker \sum$ lies in a hyperplane and is a lattice of full rank with volume equal to the regulator R_{K^+}. Thus we have $\log \iota C_{K^+} \subset \log \iota \mathcal{O}_{K^+}^\times$ is a sublattice.

To show that C_{K^+} has finite index in the unit group it suffices to show that the volume of C_{K^+}, equal to the regulator R of the elements $\log \iota \xi_a$, is nonzero in which case

$$[\mathcal{O}_{K^+}^\times : C_{K^+}] = R/R_{K^+}$$

since $\log \iota$ eliminates only ± 1 from both groups.

To compute this regulator recall that $G_{K^+/\mathbb{Q}} \cong G_{K/\mathbb{Q}}/\{\pm 1\}$ where $G_{K/\mathbb{Q}} \cong (\mathbb{Z}/p^m\mathbb{Z})^\times$. Thus writing σ_a for the element corresponding to $a \in (\mathbb{Z}/p^m\mathbb{Z})^\times$, $G_{K^+/\mathbb{Q}} = \{\sigma_a | 1 \le a < p^m/2, (a,p) = 1\}$, which is exactly one more that the number of ξ_a. Therefore we compute (we eliminate the trivial element from $G_{K^+/\mathbb{Q}}$ to obtain a square-matrix)

$$\begin{aligned} R &= |\det(\log \iota \xi_a)| \\ &= |\det(\log |\tau(\xi_a)|)_{\tau \neq 1}| \end{aligned}$$

as τ varies in $G_{K^+/\mathbb{Q}}$.

Let $\sigma_a \in G_{K^+/\mathbb{Q}} \cong G_{K/\mathbb{Q}}/\{\pm 1\}$ correspond to $a \in (\mathbb{Z}/p^m\mathbb{Z})^\times$ in which case

$$\xi_a = \frac{\sigma_a(\zeta^{-1/2}(1-\zeta))}{\zeta^{-1/2}(1-\zeta)}$$

and if we write $f(\sigma) = \log |\sigma(\zeta^{-1/2}(1-\zeta))| = \log |\sigma(1-\zeta)|$ then

$$\log |\tau(\xi_a)| = f(\tau\sigma_a) - f(\tau)$$

Thus

$$\begin{aligned}
R &= |\det(\log|\tau(\xi_a)|)_{a,\tau\neq 1}| \\
&= |\det(f(\tau\sigma_a) - f(\tau))_{a,\tau\neq 1}| \\
&= |\det(f(\tau\sigma^{-1}) - f(\tau))_{\sigma,\tau\neq 1}| \\
&= |\prod_{\chi\neq 1}\sum_{\sigma\in G}\chi(\sigma)f(\sigma)| \\
&= |\prod_{\chi\neq 1}\sum_{\sigma\in G}\chi(\sigma)\log|\sigma(\zeta^{-1/2}(1-\zeta))|| \\
&= |\prod_{\chi\neq 1}\sum_{1\leq a<p^m/2,(a,p)=1}\chi(a)\log|1-\zeta^a|| \\
&= |\prod_{\chi\neq 1}\frac{1}{2}\sum_{1\leq a\leq p^m}\chi(a)\log|1-\zeta^a||
\end{aligned}$$

where the last line comes from the fact that $\log|1-\zeta^a| = \log|1-\zeta^{-a}|$.

Let $\chi\neq 1$ with conductor $f_\chi = p^k$. Then

$$\begin{aligned}
\sum_{a=1}^{p^m}\chi(a)\log|1-\zeta^a| &= \sum_{b=1}^{p^k}\sum_{a\equiv b \pmod{p^k}}\chi(a)\log|1-\zeta^a| \\
&= \sum_{b=1}^{p^k}\chi(b)\sum_{a\equiv b \pmod{p^k}} log|1-\zeta^a| \\
&= \sum_{b=1}^{p^k}\chi(b)\log|\prod_{a\equiv b \pmod{p^k}}(1-\zeta^a)| \\
&= \sum_{b=1}^{p^k}\chi(b)\log|\prod(1-\zeta^{b+p^k c})| \\
&= \sum_{b=1}^{p^k}\chi(b)\log|1-\zeta^{bp^{m-k}}| \\
&= \sum_{b=1}^{p^k}\chi(b)\log|1-\zeta_{f_\chi}^b| \\
&= -\frac{f_{\overline{\chi}}L(\overline{\chi},1)}{\tau(\overline{\chi})} \\
&= -\tau(\chi)L(\overline{\chi},1)
\end{aligned}$$

since χ is even as K^+ is totally real.

Finally (write $n = p^{m-1}(p-1)/2 = [K^+ : \mathbb{Q}]$)

$$\begin{aligned}
[\mathcal{O}_{K^+}^\times : C_{K^+}] &= \frac{R}{R_{K^+}} \\
&= \frac{\prod_{\chi\neq 1} 2^{-1}\tau(\chi)L(\overline{\chi},1)}{R_{K^+}} \\
&= \frac{2^{1-n}\sqrt{d_K}\prod_{\chi\neq 1} L(\chi,1)}{R_{K^+}} \\
&= \frac{2^{1-n}\sqrt{d_K}\lim_{s\to 1}(\zeta_K(s)/\zeta(s))}{R_{K^+}} \\
&= \frac{2^{1-n}\sqrt{d_K}\frac{2^n(2\pi)^0 h_{K^+}R_{K^+}}{w_{K^+}\sqrt{d_K}}}{R_{K^+}} \\
&= h_{K^+}
\end{aligned}$$

using the analytic class number formula. □