Introduction to Algebraic Number Theory
Lecture 31

Andrei Jorza

13 A peculiar integral

This section is due to David Speyer.

We will compute

$$\int_0^1 \frac{\log(1+x^{2+\sqrt{3}})dx}{1+x} = \frac{\pi^2}{12}(1-\sqrt{3}) + \log(2)\log(1+\sqrt{3})$$

(13.1) Let $F(\alpha) = \int_0^1 \frac{\log(1+x^\alpha)dx}{1+x}$. Using integration by parts get

$$\begin{aligned} F(\alpha) &= \int_0^1 \log(1+x^\alpha) d\log(1+x) \\ &= (\log 2)^2 - F(\alpha^{-1}) \end{aligned}$$

But also

$$\begin{aligned} F(\alpha) &= \int_0^1 \sum_{k,n\geq 1} \frac{(-1)^{k-1}x^{k-1}(-1)^{n-1}x^{\alpha n}}{n} \\ &= \sum_{k,n\geq 1} \frac{(-1)^{n+k}}{n(n\alpha+k)} \end{aligned}$$

and so

$$\begin{aligned} F(\alpha) - F(\alpha^{-1}) &= \sum_{k,n\geq 1} \frac{(-1)^{n+k}}{n}\left(\frac{1}{n\alpha+k} - \frac{1}{n\alpha^{-1}+k}\right) \\ &= (\alpha^{-1}-\alpha)\sum_{k,n\geq 1} \frac{(-1)^{n+k}}{n^2+(\alpha+\alpha^{-1})nk+k^2} \end{aligned}$$

(13.2) For $\alpha = 2+\sqrt{3}$ this gives

$$F(\alpha) - F(\alpha^{-1}) = -2\sqrt{3}\sum_{k,n\geq 1} \frac{(-1)^{n+k}}{n^2+4kn+k^4}$$

which, under the change of variables $m = k+2n$ can be rewritten as

$$F(\alpha) - F(\alpha^{-1}) = -2\sqrt{3}\sum_{m>2n>0} \frac{(-1)^{m+n}}{m^2-3n^2}$$

Recall the maps $K^\times \stackrel{\iota}{\to} \mathbb{R}^2 \stackrel{\log}{\to} \mathbb{R}^2 \stackrel{\sum}{\to} \mathbb{R}$ taking $\mathcal{O}_K^\times$ to the lattice $\log\iota\mathcal{O}_K^\times = (R_K, -R_K)\mathbb{Z} \subset \Delta = (1,-1)\mathbb{R} \subset \mathbb{R}^2$ where $R_K = \log(2+\sqrt{3})$ is the regulator since $\mathcal{O}_K^\times = \pm(2+\sqrt{3})^{\mathbb{Z}}$. As in the proof of the

Dirichlet unit theorem the region $\mathcal{D} = \{y < x \leq y + 2R_K\}$ is a fundamental domain of representative for $\mathbb{R}^2/\log \iota\mathcal{O}_K^\times$ and the preimage of this under log gives the region $\{(x,y) \in \mathbb{R}^2 | |y| < |x| \leq |y|(2+\sqrt{3})^2\}$. Writing $(x,y) = \iota(m + n\sqrt{3})$, i.e., $x = m + n\sqrt{3}$ and $y = m - n\sqrt{3}$ the condition $m \geq 2n > 0$ translates into $xy > 0$ and $(x,y) \in \log^{-1}\mathcal{D}$ in other words $(x,y) \in \log^{-1}\mathcal{D} \cap \iota(\mathcal{O}_K^\times)$ with $N(x,y) > 0$.

Let $\mathcal{O}_K^\pm \subset \mathcal{O}_K$ be where the norm has sign $\pm$. Note that if $u = \pm(2+\sqrt{3})^k$ and $x = m + n\sqrt{3}$ if we define

$$\sigma(m + n\sqrt{3}) = (-1)^{m+n}$$

then

$$\sigma(xu) = \sigma(x)$$
$$N_{K/\mathbb{Q}}(\upsilon) = N_{K/\mathbb{Q}}(x)$$

the latter because all units in $\mathcal{O}_K$ have norm $+1$, by their classification. This states that σ and $N_{K/\mathbb{Q}}$ are well-defined independent of any translates by $\mathcal{O}_K^\times$ so in fact

$$\sum_{x \in \log^{-1}\mathcal{D}, N(x)>0} \frac{\sigma(x)}{N(x)} = \sum_{x \in \mathcal{O}_K^+/\mathcal{O}_K^\times} \frac{\sigma(x)}{N(x)}$$

under the identification $\log^{-1}\mathcal{D} = \mathcal{O}_K - 0/\mathcal{O}_K^\times$, and this is simply

$$\begin{aligned}
\sum_{x \in \log^{-1}\mathcal{D}, N(x)>0} \frac{\sigma(x)}{N(x)} &= \sum_{m \geq 2n > 0} \frac{\sigma(m+n\sqrt{3})}{N_{K/\mathbb{Q}}(m+n\sqrt{3})} \\
&= \sum_{m > 2n > 0} \frac{\sigma(m+n\sqrt{3})}{N_{K/\mathbb{Q}}(m+n\sqrt{3})} + \sum_{n \geq 1} \frac{(-1)^n}{n^2} \\
&= \sum_{m > 2n > 0} \frac{(-1)^{m+n)}}{m^2 - 3n^2} - \frac{\pi^2}{12}
\end{aligned}$$

because

$$\begin{aligned}
\sum_{n \geq 1} \frac{(-1)^n}{n^2} &= \sum_{n \in 2\mathbb{Z}, n \geq 1} \frac{1}{n^2} - \sum_{n \in 2\mathbb{Z}+1, n \geq 1} \frac{1}{n^2} \\
&= 2 \sum_{n \in 2\mathbb{Z}, n \geq 2} \frac{1}{n^2} - \sum_{n \geq 1} \frac{1}{n^2} \\
&= 2 \sum_{k \geq 1} \frac{1}{(2k)^2} - \zeta(2) \\
&= \zeta(2)/2 - \zeta(2) \\
&= -\frac{\pi^2}{12}
\end{aligned}$$

(13.3) So we only need to compute

$$\sum_{x \in \mathcal{O}_K^+/\mathcal{O}_K^\times} \frac{\sigma(x)}{N(x)}$$

Let $\chi(x) = \operatorname{sign} N_{K/\mathbb{Q}}(x)$ in which case we see that $\chi(xu) = \chi(x)$ for $x \in \mathcal{O}_K - 0$ and $u \in \mathcal{O}_K^\times$ from the equality of norms. Then to count only $x \in \mathcal{O}_K^+$ we note that $(1 + \chi(x))/2$ is 1 if $x \in \mathcal{O}_K^+$ and 0 if $x \in \mathcal{O}_K^-$ and so

$$\sum_{x \in \mathcal{O}_K^+/\mathcal{O}_K^\times} \frac{\sigma(x)}{N(x)} = \frac{1}{2} \sum_{x \in \mathcal{O}_K - 0/\mathcal{O}_K^\times} \frac{\sigma(x)(1 + \chi(x))}{|N_{K/\mathbb{Q}}(x)|}$$

where the absolute value appears because the only terms that show up in the sum are those where the norm is positive.

(13.4) Since σ, $N_{K/\mathbb{Q}}$ and χ do not change upon multiplication by units we can rewrite this as a sum over ideals as the class number of $\mathbb{Q}(\sqrt{3})$ is 1. Indeed

$$\begin{aligned}\frac{1}{2}\sum_{x\in\mathcal{O}_K-0/\mathcal{O}_K^\times}\frac{\sigma(x)(1+\chi(x))}{|N_{K/\mathbb{Q}}(x)|} &= \frac{1}{2}\sum_{I\neq 0}\frac{\sigma(I)(1+\chi(I))}{|N_{K/\mathbb{Q}}(I)|}\\ &= \frac{1}{2}\sum_{I\neq 0}\frac{\sigma(I)(1+\chi(I))}{||I||}\end{aligned}$$

where $\sigma(I), \chi(I)$ and $N_{K/\mathbb{Q}}(I)$ are defined by $\sigma(x), \chi(x)$ and $N_{K/\mathbb{Q}}(x)$ for any generator x of I and $|N_{K/\mathbb{Q}}(I)| = I$.

(13.5) We remark that $\sigma(x) = 1$ if and only if $\varpi = 1+\sqrt{3} \mid x$. Indeed,

$$\frac{m+n\sqrt{3}}{1+\sqrt{3}} = \frac{3n-m}{2} + \frac{m-n}{2}\sqrt{3}$$

is an algebraic integer if and only if $m+n$ is even, i.e., if $\sigma(m+n\sqrt{3}) = 1$. Thus

$$\begin{aligned}U(s) &= \sum_{I\neq 0}\frac{\sigma(I)}{||I||^s}\\ &= \sum_{\varpi\mid I}\frac{1}{||I||^s} - \sum_{\varpi\nmid I}\frac{1}{||I||^s}\\ &= \sum_{n\geq 1}\sum_{v_\varpi(I)=n}\frac{1}{||I||^s} - \sum_{\varpi\nmid I}\frac{1}{||I||^s}\\ &= \sum_{n\geq 1}\sum_{\varpi\nmid J}\frac{1}{2^{ns}||J||^2} - \sum_{\varpi\nmid I}\frac{1}{||I||^s}\\ &= \left(-1+\frac{1}{2^s}+\frac{1}{4^s}+\cdots\right)\sum_{\varpi\nmid I}\frac{1}{||I||^s}\\ &= \left(-2+\frac{1}{1-2^{-s}}\right)\prod_{\mathfrak{p}\neq\varpi}\left(1-\frac{1}{||\mathfrak{p}||^s}\right)^{-1}\\ &= (-1+2^{1-s})\prod_{\mathfrak{p}}\left(1-\frac{1}{||\mathfrak{p}||^s}\right)^{-1}\\ &= (-1+2^{1-s})\zeta_K(s)\end{aligned}$$

where in the sum corresponding to $v_\varpi(I) = n$ we write $I = \varpi^n J$ with $\varpi \nmid J$. Here the last product is over all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$.

Thus

$$\begin{aligned}
\sum_{I\neq 0}\frac{\sigma(I)}{||I||} &= U(1)\\
&= \lim_{s\to 1}\frac{-1+2^{1-s}}{s-1}\lim_{s\to 1}(s-1)\zeta_K(s)\\
&= -\log(2)\frac{2^2R_K}{2\sqrt{12}}\\
&= -\frac{\log(2)\log(2+\sqrt{3})}{\sqrt{3}}
\end{aligned}$$

where the last line is the analytic class number formula.

(13.6) Next we compute

$$\begin{aligned}
V(s) &= \sum_{I\neq 0}\frac{\sigma(I)\chi(I)}{||I||^2}\\
&= \sum_{n\geq 1}\sum_{\varpi\nmid J}\frac{\chi(\varpi^nJ)}{2^{ns}||J||^s} - \sum_{\varpi\nmid I}\frac{\chi(I)}{||I||^s}
\end{aligned}$$

but now $\chi(\varpi^nJ)=\chi(\varpi)^n\chi(J)=(-1)^n\chi(J)$ and so

$$\begin{aligned}
V(s) &= \left(-1-\frac{1}{2^s}+\frac{1}{4^s}-\frac{1}{8^s}\cdots\right)\sum_{\varpi\nmid I}\frac{\chi(I)}{||I||^s}\\
&= \frac{-1-2^{1-s}}{1+2^{-s}}\prod_{\mathfrak{p}\neq\varpi}\left(1-\frac{\chi(\mathfrak{p})}{||\mathfrak{p}||^s}\right)^{-1}\\
&= \frac{-1-2^{1-s}}{1+2^{-s}}(1+2^{-s})\prod_{\mathfrak{p}}\left(1-\frac{\chi(\mathfrak{p})}{||\mathfrak{p}||^s}\right)^{-1}\\
&= (-1-2^{1-s})L(\chi,s)
\end{aligned}$$

because $\chi(\varpi)=-1$ and here we denoted

$$L(\chi,s)=\prod_{\mathfrak{p}}\left(1-\frac{\chi(\mathfrak{p})}{||\mathfrak{p}||^s}\right)^{-1}$$

From the homework

$$L(\chi,s)=L(\chi_4,s)\,L(\chi_3,s)$$

where $\chi_d(x)=\left(\frac{x}{d}\right)$ and so to compute $V(1)=\sum_{I\neq 0}\frac{\sigma(I)\chi(I)}{||I||}$ we only need to compute the two special values. But characters χ_4 and χ_3 are odd as -1 is not a quadratic residue mod 4 or 3 and so

$$L(\chi_4,1)=\frac{\pi i\tau(\chi_4)B_{1,\overline{\chi}_4}}{f_{\chi_4}}=\frac{\pi i\cdot 2i\cdot -\frac{1}{2}}{4}=\frac{\pi}{4}$$

and

$$L(\chi_3,1)=\frac{\pi i\tau(\chi_3)B_{1,\overline{\chi}_3}}{3}=\frac{\pi i\cdot i\sqrt{3}\cdot -\frac{1}{3}}{3}=\frac{\pi}{3\sqrt{3}}$$

and so

$$L(\chi,1)=\frac{\pi^2}{12\sqrt{3}}$$

giving

$$V(1) = -2L(\chi, 1) = -\frac{\pi^2}{6\sqrt{3}}$$

(13.7) Putting everything together we get

$$\frac{1}{2}\sum_{I\neq 0}\frac{\sigma(I)(1+\chi(I))}{||I||} = \frac{1}{2}\left(-\frac{\log(2)\log(2+\sqrt{3})}{\sqrt{3}} - \frac{\pi^2}{6\sqrt{3}}\right)$$

and so

$$F(2+\sqrt{3}) - F(2-\sqrt{3}) = -2\sqrt{3}\left(\frac{\pi^2}{12} + \frac{1}{2}\left(-\frac{\log(2)\log(2+\sqrt{3})}{\sqrt{3}} - \frac{\pi^2}{6\sqrt{3}}\right)\right)$$

Together with $F(2+\sqrt{3}) + F(2-\sqrt{3}) = (\log(2))^2$ this gives

$$F(2+\sqrt{3}) = \frac{(\log 2)^2}{2} - \frac{\pi^2\sqrt{3}}{12} + \frac{\pi^2}{12} + \frac{\log(2)\log(2+\sqrt{3})}{2} = \frac{\pi^2}{12}(1-\sqrt{3}) + \log(2)\log(1+\sqrt{3})$$

as $2(2+\sqrt{3}) = (1+\sqrt{3})^2$.