Algebraic Number Theory Notes
Lecture 38

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We continue from last time: $q = p^r$ for p a prime, $K = \mathbb{F}_q$, and $\mathcal{C}$ a smooth projective curve over K. Recall that

$$\mathrm{Div}(\mathcal{C}/K) = \left\{ \sum (n_p[p]) : \forall \sigma \in G_K, n_p = n_{\sigma(p)} \right\}$$

and that for all $n \in \mathbb{N}$,

$$|\{D \in \mathrm{Div}(\mathcal{C}/K) : \|D\| < n\}| < \infty$$

where $\|D\| = q^{\deg(D)}$. Recall also that $\mathrm{Pic}^0(\mathcal{C}/K)$ is finite with $h_{\mathcal{C}}$ many elements. We define the define the following zeta function associated to $\mathcal{C}$:

$$\zeta_{\mathcal{C}}(s) = \sum_{\substack{D \geq 0 \\ D \in \mathrm{Div}(\mathcal{C}/K)}} \frac{1}{\|D\|^s}.$$

Our goal is to prove the following theorem about $\zeta_{\mathcal{C}}$:

Theorem 1. *1. There is a polynomial $P(z) \in \mathbb{Z}[z]$ of degree $2g$ satisfying*

$$P(z) = q^g z^{2g} P\left(\frac{1}{qz}\right)$$

and such that

$$\zeta_{\mathcal{C}}(s) = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})}.$$

2. $P(0) = 1$ and $P(1) = h_{\mathcal{C}}$ and so with part 1. we get immediately that

$$\lim_{s \to \infty} s\zeta_{\mathcal{C}}(s) = \frac{h_{\mathcal{C}}}{(q-1)(\log(q))}.$$

The proof extensively uses Riemann-Roch:

$$\mathcal{L}(D) = \{f \in \overline{K}(\mathcal{C}) : \mathrm{div}(f) \geq -D\}$$

and

$$l(D) - l(K_{\mathcal{C}} - D) = \deg(D) - g + 1.$$

In particular, if $\deg(D) > 2g - 2$ then

$$\dim(\mathcal{L}(D)) = \deg(D) - g + 1. \tag{1}$$

Proof of Theorem 1. Observe that we can write $\zeta_{\mathcal{C}}(s)$ as follows:

$$\zeta_{\mathcal{C}}(s) = \sum_{D\geq 0} q^{-\deg(D)s} = \sum_{d=0}^{\infty} \sum_{\substack{D\geq 0 \\ \deg(D)=d}} q^{-ds}.$$

Recall from last time that there exists $u \in \operatorname{Div}(\mathcal{C}/K)$ with $\deg(u) = 1$ and so we can rewrite the above equation as

$$\zeta_{\mathcal{C}}(s) = \sum_{d=0}^{\infty} \sum_{\substack{D\geq 0 \\ \deg(D-du)=0}} q^{-ds}.$$

Since $D - du \in \operatorname{Div}^0(\mathcal{C}/K)$ and $\operatorname{Div}^0(\mathcal{C}/K)$ projects onto $\operatorname{Pic}^0(\mathcal{C}/K) = \{\alpha_1, \ldots, \alpha_{h_{\mathcal{C}}}\}$ which is finite, we can futher rewrite this as

$$\begin{aligned}
\zeta_{\mathcal{C}}(s) &= \sum_{d\geq 0} \sum_{\alpha\in\operatorname{Pic}^0(\mathcal{C}/K)} \sum_{\substack{D\geq 0 \\ D-du=\alpha\in\operatorname{Pic}^0}} q^{-ds} \\
&= \sum_{d\geq 0} q^{-ds} \underbrace{\left(\sum_{\alpha\in\operatorname{Pic}^0(\mathcal{C}/K)} \sum_{\substack{D\geq 0 \\ D-du=\alpha\in\operatorname{Pic}^0}} 1 \right)}_{\text{Summing over the representative classes } \alpha\in\operatorname{Pic}^0}.
\end{aligned}$$

Now, let's try to compute $|\{D \geq 0 : D - du = \alpha \in \operatorname{Pic}^0(\mathcal{C}/K)\}|$:
If $D - du = \alpha \in \operatorname{Pic}^0(\mathcal{C}/K)$, then

$$D - du = \alpha + \operatorname{div}(f)$$

for some function f. Then $D = du + \alpha + \operatorname{div}(f)$ and so

$$\begin{aligned}
|\{D \geq 0 : D - du = \alpha \in \operatorname{Pic}^0\}| &= |\{\operatorname{div}(f) : \underbrace{du + \alpha + \operatorname{div}(f)}_{D} \geq 0\}| \\
&= |\{\operatorname{div}(f) : \operatorname{div}(f) \geq -(du + \alpha)\}| \\
&= |\{\operatorname{div}(f) : f \in \underbrace{\mathcal{L}(du + \alpha)}_{\text{considered as a vector space over } K}\}|.
\end{aligned}$$

Recall that the map $\operatorname{div} : \overline{K}(\mathcal{C})^\times \to \operatorname{Div}(\mathcal{C}/K)$ has kernel $\ker(\operatorname{div}) = \overline{K}^\times$ and so, by the first isomorphism theorem and the fact that $\mathcal{L}(du + \alpha)$ is an $l(du + \alpha)$-dimensional K-vector space, we get that

$$\begin{aligned}
|\{\operatorname{div}(f) : f \in \mathcal{L}(du + \alpha)\}| &= \frac{|K^{l(du-\alpha)} - \{0\}|}{|K - \{0\}|} \\
&= \frac{q^{l(du+\alpha)} - 1}{q - 1}
\end{aligned}$$

and we can use Riemann-Roch to find $l(du + \alpha)$. Using this fact, we can write

$$\zeta_{\mathcal{C}}(s) = \sum_{D\geq 0} q^{-ds} \left(\sum_{\alpha\in\operatorname{Pic}^0(\mathcal{C}/K)} \sum_{\substack{D\geq 0 \\ D-du=\alpha\in\operatorname{Pic}^0}} 1 \right)$$

$$= \sum_{d\geq 0} q^{-ds} \sum_{\alpha\in \mathrm{Pic}^0(\mathcal{C}/K)} \frac{q^{l(du+\alpha)}-1}{q-1}$$

and by Riemann-Roch,

$$\begin{aligned}
\zeta_{\mathcal{C}}(s) &= \sum_{d=0}^{2g-2}\left(q^{-ds}\sum_{\alpha\in \mathrm{Pic}^0(\mathcal{C}/K)}\frac{q^{l(du+\alpha)}-1}{q-1}\right) + \sum_{d\geq 2g-1} q^{-ds} \sum_{\alpha\in \mathrm{Pic}^0(\mathcal{C}/K)} \underbrace{\frac{q^{d-g+1}-1}{q-1}}_{\text{Independent of}\alpha} \\
&= \sum_{d=0}^{2g-2}\left(q^{-ds}\sum_{\alpha\in \mathrm{Pic}^0(\mathcal{C}/K)}\frac{q^{l(du+\alpha)}-1}{q-1}\right) + \frac{h_{\mathcal{C}}}{q-1}\sum_{d\geq 2g-1}\left(q^{1-g}q^{(1-s)d}-q^{-sd}\right) \\
&= \sum_{d=0}^{2g-2}\left(q^{-ds}\sum_{\alpha\in \mathrm{Pic}^0(\mathcal{C}/K)}\frac{q^{l(du+\alpha)}-1}{q-1}\right) + \frac{h_{\mathcal{C}}}{q-1}\left(\frac{q^{1-g+(1-s)(2g-1)}}{1-q^{1-s}} - \frac{q^{-s(2g-1)}}{1-q^{-s}}\right) \\
&= \sum_{d=0}^{2g-2}\left(Z^d\sum_{\alpha\in \mathrm{Pic}^0(\mathcal{C}/K)}\frac{q^{l(du+\alpha)}-1}{q-1}\right) + \frac{h_{\mathcal{C}}Z^{2g-1}}{(1-Z)(1-qZ)}\frac{(q^g-1)-q(q^{g-1}-1)Z}{q-1}
\end{aligned}$$

where we denoted $Z = q^{-s}$. It is now clear that $P(Z) = \zeta_C(s)(1-Z)(1-qZ)$ is a polynomial of degree $2g$ and has integral coefficients.

To prove the functional equation for $P(Z)$ we rewrite the above computation as follows (again using the notation $Z = q^{-s}$):

$$\begin{aligned}
\zeta_{\mathcal{C}}(s) &= \sum_{d=0}^{2g-2}\left(q^{-ds}\sum_{\alpha\in \mathrm{Pic}^0(\mathcal{C}/K)}\frac{q^{l(du+\alpha)}-1}{q-1}\right) + \frac{h_{\mathcal{C}}}{q-1}\sum_{d\geq 2g-1}\left(q^{1-g}q^{(1-s)d}-q^{-sd}\right) \\
&= \sum_{d=0}^{2g-2}\left(q^{-ds}\sum_{\alpha\in \mathrm{Pic}^0(\mathcal{C}/K)}\frac{q^{l(du+\alpha)}}{q-1}\right) + \frac{h_{\mathcal{C}}}{q-1}\sum_{d\geq 2g-1}q^{1-g}q^{(1-s)d} - \frac{h_{\mathcal{C}}}{q-1}\sum_{d\geq 0}q^{-sd} \\
&= \sum_{\alpha\in \mathrm{Pic}^0(\mathcal{C}/K)}\sum_{d=0}^{2g-2}Z^d\frac{q^{l(du+\alpha)}}{q-1} + \frac{h_{\mathcal{C}}}{q-1}\left(\frac{q^{1-g+(1-s)(2g-1)}}{1-q^{1-s}} - \frac{1}{1-q^{-s}}\right) \\
&= \sum_{\alpha\in \mathrm{Pic}^0(\mathcal{C}/K)}\sum_{d=0}^{2g-2}Z^d\frac{q^{l(du+\alpha)}}{q-1} + \frac{h_{\mathcal{C}}}{q-1}\left(\frac{q^gZ^{2g-1}}{1-qZ} - \frac{1}{1-Z}\right) \\
&= \frac{(1-Z)(1-qZ)\sum_{\alpha\in \mathrm{Pic}^0(\mathcal{C}/K)}\sum_{d=0}^{2g-2}Z^d\frac{q^{l(du+\alpha)}}{q-1} + \frac{h_{\mathcal{C}}}{q-1}(q^gZ^{2g-1}(1-Z)-(1-qZ))}{(1-Z)(1-qZ)}
\end{aligned}$$

We now study the functional equation satisfied by the numerator:

$$P(z) = (1-qz)(1-z)\left(\sum_{\alpha\in \mathrm{Pic}^0(\mathcal{C}/K)}\sum_{d=0}^{2g-2}z^d\frac{q^{l(du+\alpha)}}{q-1}\right) + \left(\frac{h_{\mathcal{C}}}{q-1}\right)\left(q^gz^{2g-1}(1-z)-(1-qz)\right)$$

which, as we remarked above, is a polynomial $P(z)\in\mathbb{Z}[z]$ of degree $2g$. Now,

$$P(0) = \lim_{s\to\infty} P(q^{-s})$$

$$
\begin{aligned}
&= \lim_{s\to\infty} \zeta_{\mathcal{C}}(s) \underbrace{(1-q^{-s})(1-q^{1-s})}_{\to 1 \text{ as } s\to\infty} \\
&= \lim_{s\to\infty} \sum_{\substack{D\geq 0 \\ D\in \mathrm{Div}(\mathcal{C}/K)}} \frac{1}{\|D\|^s} \\
&= \lim_{s\to\infty} \left(1 + \sum_{\substack{D>0 \\ D\in \mathrm{Div}(\mathcal{C}/K)}} \frac{1}{\|D\|^s} \right) \\
&= 1
\end{aligned}
$$

which is what we wanted. Furthermore, it is clear that

$$P(1) = \frac{h_{\mathcal{C}}}{q-1}(-(1-q)) = h_{\mathcal{C}}.$$

At this point, we only need to show that $P(z)$ satisfies

$$P(z) = q^g z^{2g} P\left(\frac{1}{qz}\right). \tag{2}$$

We show this for $P(z)$ by showing it is true for fragments: Let

$$B(z) = \left(\frac{h_{\mathcal{C}}}{q-1}\right) \left(q^g z^{2g-1}(1-z) - (1-qz)\right),$$

the last part of $P(z)$. We want to show that $B(z)$ satisfies 2. Substituting, we get

$$
\begin{aligned}
B\left(\frac{1}{qz}\right) &= \left(\frac{h_{\mathcal{C}}}{q-1}\right) \left(q^g (qz)^{-2g+1} \left(\frac{qz-1}{qz}\right) - \left(\frac{z-1}{z}\right)\right) \\
&= \left(\frac{h_{\mathcal{C}}}{q-1}\right) \left(q^{-g+1} z^{-2g+1} \left(\frac{qz-1}{qz}\right) - \left(\frac{z-1}{z}\right)\right) \\
&= \left(\frac{h_{\mathcal{C}}}{q-1}\right) q^{-g} z^{-2g} \left(qz \left(\frac{qz-1}{qz}\right) - q^g z^{2g} \left(\frac{z-1}{z}\right)\right) \\
&= q^{-g} z^{-2g} \left(\frac{h_{\mathcal{C}}}{q-1}\right) \left((qz-1) - q^g z^{2g-1}(z-1)\right) \\
&= q^{-g} z^{-2g} \left(\frac{h_{\mathcal{C}}}{q-1}\right) \left(-(1-qz) + q^g z^{2g-1}(1-z)\right) \\
&= q^{-g} z^{-2g} B(z),
\end{aligned}
$$

as required. For each $\alpha \in \mathrm{Pic}^0(\mathcal{C}/K)$, let

$$A_\alpha(z) = \frac{(1-qz)(1-z)}{q-1} \sum_{d=0}^{2g-2} q^{l(du+\alpha)} z^d.$$

We claim that there is $\beta \in \mathrm{Pic}^0(\mathcal{C}/K)$ such that for each $\alpha \in \mathrm{Pic}^0(\mathcal{C}/K)$, we have

$$A_\alpha(z) = q^g z^{2g} A_{(\beta-\alpha)}\left(\frac{1}{qz}\right).$$

If this claim holds, then since the map $\alpha \mapsto (\beta - \alpha)$ is a permutation of $\mathrm{Pic}^0(\mathcal{C}/K)$, the sum

$$\sum_{\alpha\in \mathrm{Pic}^0(\mathcal{C}/K)} A_\alpha(z)$$

satisfies 2 and so we will be done. Substituting as we did for $B(z)$,

$$\begin{aligned} q^g z^{2g} A_\alpha\left(\frac{1}{qz}\right) &= \frac{q^g z^{2g}}{q-1}\left(\frac{qz-1}{qz}\right)\left(\frac{z-1}{z}\right)\sum_{d=0}^{2g-2}\frac{q^{l(du+\alpha)}}{q^d z^d} \\ &= \frac{q^{g-1}z^{2g-2}}{q-1}\left(qz-1\right)\left(z-1\right)\sum_{d=0}^{2g-2}\frac{q^{l(du+\alpha)}}{q^d z^d} \\ &= \frac{\left(1-qz\right)\left(1-z\right)}{q-1}\sum_{d=0}^{2g-2} q^{l(du+\alpha)} q^{g-1-d} z^{2g-2-d}. \end{aligned}$$

Now, setting $d = 2g-2-d$ and summing the other way around, we get

$$q^g z^{2g} A_\alpha\left(\frac{1}{qz}\right) = \frac{\left(1-qz\right)\left(1-z\right)}{q-1}\sum_{d=0}^{2g-2} q^{l((2g-2-d)u+\alpha)} q^{d-g+1} z^d. \tag{3}$$

Here, recall that $\deg(K_{\mathcal{C}}) = 2g-2$ and so $K_{\mathcal{C}} - (2g-2)u = \beta$ some fixed element in $\mathrm{Pic}^0(\mathcal{C}/K)$. By Riemann-Roch,

$$l(D) - l(K_{\mathcal{C}} - D) = \deg(D) - g + 1$$

and so for each $\alpha \in \mathrm{Pic}^0(\mathcal{C}/K)$,

$$\begin{aligned} l((2g-2-d)u+\alpha) - l(\beta + (2g-2)u - (2g-2-d)u - \alpha) &= 2g-2-d-g+1 \\ &= g-1-d. \end{aligned}$$

Therefore,

$$l((2g-2-d)u+\alpha) = l(du+\beta-\alpha) + g - 1 - d.$$

Substituting this into 3 we get

$$\begin{aligned} q^g z^{2g} A_\alpha\left(\frac{1}{qz}\right) &= \frac{\left(1-qz\right)\left(1-z\right)}{q-1}\sum_{d=0}^{2g-2} q^{l((du+(\beta-\alpha))+g-1-d} q^{d-g+1} z^d \\ &= \frac{\left(1-qz\right)\left(1-z\right)}{q-1}\sum_{d=0}^{2g-2} q^{l((du+(\beta-\alpha))} z^d \\ &= A_{(\beta-\alpha)}(z), \end{aligned}$$

which proves the claim. This completes the proof. □