Mixing Tank Separable Differential Equations Examples

When studying separable differential equations, one classic class of examples is the mixing tank problems. Here we will consider a few variations on this classic.

Example 1. A tank has pure water flowing into it at 10 l/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 l/min. Initially, the tank contains 10 kg of salt in 100 l of water.

How much salt will there be in the tank after 30 minutes?

To study such a question, we consider *the rate of change of the amount of salt in the tank*. Let S be the amount of salt in the tank at any time t. If we can create an equation relating $\frac{dS}{dt}$ to S and t, then we will have a differential equation which we can, ideally, solve to determine the relationship between S and t.

To describe $\frac{dS}{dt}$, we use the concept of *concentration*, the amount of salt per unit of volume of liquid in the tank. In this example, the inflow and outflow rates are the same, so the volume of liquid in the tank stays constant at 100 l. Hence, we can describe the concentration of salt in the tank by

$$\text{concentration of salt} = \frac{S}{100}\,\text{kg/l}.$$

Then, since mixture leaves the tank at the rate of 10 l/min, salt is leaving the tank at the rate of

$$\frac{S}{100}(10\,\text{l/min}) = \frac{S}{10}.$$

This is the rate at which salt *leaves* the tank, so

$$\frac{dS}{dt} = -\frac{S}{10}.$$

This is the differential equation we can solve for S as a function of t. Notice that since the derivative is expressed in terms of a single variable, it is the simplest form of separable differential equations, and can be solved as follows:

$$\int \frac{dS}{S} = -\int \frac{1}{10}\,dt$$

$$\ln|S| = -\frac{1}{10}t + C$$

$$S = Ce^{-\frac{1}{10}t}$$

where C is a positive constant. Note that we have used the fact that $S \geq 0$ to eliminate the absolute value symbol.

Since $S = 10$ when $t = 0$, we find $C = 10$ and finally we have

$$S = 10e^{-\frac{1}{10}t}.$$

We can see from this that as t goes to infinity, the amount of salt in the tank goes to zero. Also, after 30 minutes, there will be

$$S = 10e^{-3} = 0.49787068\,\text{kg}$$

of salt in the tank.

Example 2. A tank has pure water flowing into it at 10 l/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 l/min. Salt is added to the tank at the rate of 0.1 kg/min. Initially, the tank contains 10 kg of salt in 100 l of water.

How much salt is in the tank after 30 minutes?

Here the setup is very similar the previous example. The only difference from the previous example is the addition of 0.1 kg/min of salt to the tank. As a result, we can modify our differential equation to take this into account:

$$\frac{dS}{dt} = -\frac{S}{10} + 0.1 = -0.1S + 0.1$$

Here we see the effect of the outflow as a negative term and the addition of salt as a positive term which we sum to get the net rate of change of salt.

Yet again, this equation is clearly separable, since there is no t variable on the right hand side. We thus solve in the standard way:

$$\int \frac{dS}{-0.1S + 0.1} = \int dt$$

$$-10 \ln|-0.1S + 0.1| = t + C$$

$$-0.1S + 0.1 = Ce^{-0.1t}$$

$$S = 1 + Ce^{-0.1t}.$$

Here C may be positive or negative, depending on the initial conditions. We see that as t approaches infinity, S approaches 1 kg regardless of the initial conditions. For this example, there were initially 10 kg of salt in the tank, so we can solve for C and find C=9. Thus,

$$S = 1 + 9e^{-0.1t}.$$

After 30 minutes, there will be 1.448 kg of salt in the tank.

Unlike the previous example, the amount of salt in the tank does not go to zero as t goes to infinity: S goes to 1. Notice that if there was 1 kg of salt in the tank, then the outflow rate of salt will be (0.01 kg/l)(10 l/min) = 0.1 kg/min, which will exactly balance the inflow rate of salt.

Example 3. A tank has pure water flowing into it at 12 l/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 l/min. Initially, the tank contains 10 kg of salt in 100 l of water.

In this case, the inflow rate is greater than the outflow rate. As a result, the volume is not constant. Using the initial conditions and the flow rates, we can say that the volume V of liquid in the tank is

$$V = 100 + 2t$$

after t minutes. The concentration of salt after t minutes is then

$$\frac{S}{V} = \frac{S}{100 + 2t}$$

and the rate of change of S is

$$\frac{dS}{dt} = -\frac{S}{100 + 2t}(10\,\mathrm{l/min}) = -\frac{10S}{100 + 2t}.$$

Once again, this is a separable differential equation, and we can solve it:

$$\int \frac{dS}{S} = \int \frac{-10}{100 + 2t}\,dt$$

$$\ln S = -5\ln(100 + 2t) + C$$

$$S = C(100 + 2t)^{-5}.$$

Note we have used the fact that $S \geq 0$ and $V = 100 + 2t \geq 0$ to eliminate absolute value symbols from the equation. With the initial conditions, we can solve for C:

$$10 = C(100 + 0)^{-5}$$

$$C = 10^{11}$$

and thus

$$S = \frac{10^{11}}{(100 + 2t)^5}.$$

After 30 minutes, there will be 0.953674 kg of salt in the tank. Notice this is more than in example 1 due to the fact that the increased inflow rate dilutes the salt, and reduces the outflow rate of salt, so the amount of salt in the tank will be greater than in example 1.

Other examples. There are a number of other variations to consider. You might experiment and investigate some of these possibilities:

- Pure water in at a rate *less* than the outflow rate of mixture.
- Rather than pure water coming into the tank, water with salt dissolved in it at a specified concentration. Consider this both in the case of constant volume, and with non-constant volume.

In some of these possibilities, you may find that the differential equation you create is *not* separable. So, unfortunately, you won't be able to solve it using the methods from this course. But, it is good to practice creating the differential equations.