MOTIVIC INVARIANTS AND SINGULARITIES

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1. Igusa local zeta function, lecture 1

1.1. Preliminary aside.

Consider the equations

$$\begin{aligned} x+1 &\equiv 0 \pmod 5 & x &\equiv 4 \pmod 5 \\ x+1 &\equiv 0 \pmod{5^2} & x &\equiv 4+4\cdot 5 \equiv 24 \pmod{5^2} \\ x+1 &\equiv 0 \pmod{5^3} & x &\equiv 4+4\cdot 5+4\cdot 5^2 \equiv 124 \pmod{5^3}, \end{aligned}$$

where each solution is a lift of the previous solution. So

$$x = 4 + 4\cdot 5 + 4\cdot 5^2 + \dots$$

is a solution to

$$x+1 \equiv 0 \pmod{5^n}$$

for any $n \in \mathbb{N}$.

Another example:

$$\begin{aligned} 3x &\equiv 2 \pmod 5 & x &\equiv 4 \pmod 5 \\ 3x &\equiv 2 \pmod{5^2} & x &\equiv 4+1\cdot 5 \pmod{5^2} \\ 3x &\equiv 2 \pmod{5^3} & x &\equiv 4+1\cdot 5+3\cdot 5^2 \pmod{5^3}, \end{aligned}$$

and so on. We want to have

$$4 + 1\cdot 5 + 3\cdot 5^2 + \dots \to \frac{2}{3}.$$

1.2. Introduction to the p-adic numbers.

We will denote the p-adic numbers $\mathbb{Q}_p$, and the real numbers $\mathbb{R} = \mathbb{Q}_\infty$. We consider $\mathbb{Q}$ to be a "global" field. We have

$$\mathbb{N} \hookrightarrow \mathbb{Z} \hookrightarrow \mathbb{Q} \hookrightarrow \mathbb{Q}_p$$

for $p \in \{\text{primes}\} \cup \{\infty\}$.

Important figures:

- Kurt Hensel, 1861–1941 (1897)
- Helmut Hasse, 1889–1979 (1920)

1.3. **Local/global principle (Hasse principle).** If $\mathscr{P}$ is a suitable property, then $\mathscr{P}$ holds in $\mathbb{Q}$ if and only if $\mathscr{P}$ holds in $\mathbb{Q}_p$ for all p prime, $p = \infty$.

Example 1.1 (Hasse's thesis). The quadratic form

$$f(x_1, \dots, x_n) = a_1 x_1^2 + a_2 x_1 x_2 + \cdots + a_m x_n^2$$

Example 1.2 (The Riemann zeta function).

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}} = \prod_p \zeta_p(s).$$

1.4. **Absolute values on $\mathbb{Q}$.** An *absolute value* on $\mathbb{Q}$ is a map

$$|\cdot| : \mathbb{Q} \longrightarrow \mathbb{Q}^+$$

such that for all $x, y \in \mathbb{Q}$,

(i) $|x| \geq 0$, $|x| = 0 \iff x = 0$;
(ii) $|x \cdot y| = |x| \cdot |y|$;
(iii) $|x + y| \leq |x| + |y|$.

Up to equivalence (the same sequences converge), there are three absolute values on $\mathbb{Q}$:

$$|x|_\infty = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} \tag{1}$$

$$|x|_0 = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{2}$$

$$|x|_p = \begin{cases} 0, & x = 0 \\ \frac{1}{p^\alpha}, & x \neq 0, \ \mathrm{ord}_p(x) = \alpha, \end{cases} \tag{3}$$

where we define the *order* $\mathrm{ord}_p(x)$ to be the unique integer α such that

$$x = p^\alpha \frac{a}{b},$$

where $p \not| \, a, b$.

For the third absolute value, we have the *ultrametric property*

$$|x + y|_p \leq \max(|x|_p, |y|_p). \tag{3a}$$

1.5. **Examples with the p-adic absolute value.** Set $p = 5$. We have

$$\begin{aligned} |1000|_5 &= \left|5^3 \cdot 8\right|_5 = \frac{1}{5^3}, \\ |1001|_5 &= \left|5^0 \cdot 1001\right|_5 = 1, \\ |1002|_5 &= 1, \\ |1005|_5 &= \frac{1}{5}. \end{aligned}$$

1.6. **Motivation for the p-adic numbers.** If $x \in \mathbb{Z}$, then $|x|_p \leq 1$. Also, if $x \in \mathbb{Q}$ with no p in the denominator, then $|x|_p \leq 1$. If $x \in \mathbb{Q}$ with no p in the numerator or denominator, then $|x|_p = 1$. Finally, if $x \in \mathbb{Q} \setminus \mathbb{Z}$ with p in the denominator, then $|x|_p > 1$.

1.7. **Completion.** Recall that the real numbers $\mathbb{R}$ are constructed as the set of all equivalence classes of Cauchy sequences of rational numbers.

A sequence $\{a_n\}$ is a *Cauchy sequence* if for all $\varepsilon > 0$, there exists N_ε such that

$$|a_n - a_m|_\infty < \varepsilon$$

for all $n, m > N_\varepsilon$.

Example 1.3 (Cauchy sequences for $|\cdot|_\infty$).

$$\begin{aligned} 0 &= \{0,0,0,\dots\} \sim \{.1,.01,.001,\dots\}, \\ 1 &= \{1,1,1,\dots\} \sim \{.9,.99,.999,\dots\}. \end{aligned}$$

We can also complete with respect to the p-adic numbers by replacing $|\cdot|_\infty$ with $|\cdot|_p$ in the above definition.

Example 1.4 (Cauchy sequences for $|\cdot|_p$).

$$\begin{aligned} 0 &= \{0,0,0,\dots\} \sim \left\{p,p^2,p^3,\dots\right\}, \\ 1 &= \{1,1,1,\dots\} \sim \left\{1+p,1+p^2,1+p^3,\dots\right\}. \end{aligned}$$

1.8. **Uniqueness of p-adic expansion.**

Theorem 1.5. *Given $x \in \mathbb{Q}$, we can uniquely write*

$$x = p^\alpha\left(a_0 + a_1 p + a_2 p^2 + \dots\right) = \left\{p^\alpha a_0, p^\alpha a_0 + p^{\alpha+1} a_1, \dots\right\},$$

where $0 \le a_i \le p-1$ and $a_0 \ne 0$.

We write $\mathbb{Z}_p$ for the *p-adic integers*, the completion of $\mathbb{Z}$ with respect to $|\cdot|_p$. In other words, these are the p-adic numbers with $\alpha \ge 0$ in the above theorem.

We can visualize the p-adic integers by placing them in nested circles based on congruences modulo p.[1]

1.9. **Sketch of integration.** We can define a measure on $\mathbb{Z}_p$ as follows: $m(\mathbb{Z}_p) = 1$, and in general,

$$m(a + p^n \mathbb{Z}_p) = \frac{1}{p^n}.$$

In other words, each of the p "balls" at a given layer has the same measure, i.e.,

$$m(p\mathbb{Z}_p) = \frac{1}{p}.$$

This is the correct definition in order to obtain a translation-invariant measure.

2. Tropical geometry, lecture 1

Reference: Macagan and Sturmfels, *Introduction to Tropical Geometry*.

[1]See `https://en.wikipedia.org/wiki/File:3-adic_integers_with_dual_colorings.svg`

2.1. The tropical semiring.

The *tropical semiring* is the set $\mathbb{R} \cup \{\infty\} = \overline{\mathbb{R}}$ with the following operations:

- tropical addition $\oplus$ is the minimum;
- tropical multiplication $\odot$ is classical addition.

Some properties:

(1) Tropical addition and multiplication are commutative and associative.
(2) The additive identity is ∞:
$$a \oplus \infty = \min(a, \infty) = a$$
for all $a \in \overline{\mathbb{R}}$.
(3) The multiplicative identity is 0:
$$a \odot 0 = a$$
for all $a \in \overline{\mathbb{R}}$.
(4) Distributive law:
$$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$$
for all $a, b, c \in \overline{\mathbb{R}}$.
(5) There are no additive inverses, which is why $\overline{\mathbb{R}}$ is a "semiring" instead of a ring. (There are multiplicative inverses for all numbers other than ∞.)

Example 2.1.
$$\begin{aligned} 2 \odot (3 \oplus 4) &= 2 \odot 3 = 5 \\ &= 2 \odot 3 \oplus 2 \odot 4 = 5 \oplus 6 = 5. \end{aligned}$$

2.2. Graphs of polynomials.

Write $x^n = x \odot \ldots \odot x$, as usual.

Example 2.2. What is the graph of $x^2 \oplus x \oplus 1$? We have
$$x^2 \oplus x \oplus 1 = \min(x \odot x, x, 1).$$
This factors into two linear polynomials:
$$(x \oplus 0) \odot (x \oplus 1) = x^2 \oplus 0 \odot x \oplus 1 \odot x \oplus 1 = x^2 \oplus x \oplus 1.$$
The "roots" are where the function isn't linear.

Example 2.3 (A double root). In the case of $x^2 \oplus 1 \odot x \oplus 1$, the $1 \odot x$ term is never the minimum, so it doesn't appear in the graph.

Since the slope changes at $x = \frac{1}{2}$, but changes by 2, let's call that a *double root*. But
$$\left(x \oplus \frac{1}{2}\right)^2 = x^2 \oplus \frac{1}{2} \odot x \oplus \frac{1}{2} \odot x \oplus 1 = x^2 \oplus \frac{1}{2} \odot x \oplus 1.$$
So these two polynomials
$$x^2 \oplus 1 \odot x \oplus 1, x^2 \oplus \frac{1}{2} \odot x \oplus 1$$
define the same function $\overline{\mathbb{R}} \longrightarrow \overline{\mathbb{R}}$.

Remark 2.4. Therefore, while we cannot always factor, we can always get a factorization that defines the same function.

2.3. **The tropical fundamental theorem.**

Theorem 2.5 (Tropical fundamental theorem of algebra)**.** *For any tropical polynomial of degree d*

$$a_d \odot x^d \oplus \ldots \oplus a_0 \qquad (a_i \in \overline{\mathbb{R}}),$$

there is a unique product of d linear factors

$$a_d \odot (x \oplus r_1) \odot \ldots \odot (x \oplus r_d)$$

which defines the same function $\overline{\mathbb{R}} \longrightarrow \overline{\mathbb{R}}$.

The constants $r_1, \ldots, r_d$ correspond to points where the function is not linear; the multiplicity of the root is the amount by which the slope changes.

The proof is left as an exercise.

2.4. **Polynomials in several variables.** Q: What about polynomials in more variables?

Example 2.6. Consider $x \oplus y \oplus 0$: now the function fails to be linear on the line $x = y$ for $x, y < 0$ and on the positive x and y axes.

Definition 2.7. Given a tropical polynomial f in n variables, the corresponding *tropical hypersurface* is the set of points where f is not linear, i.e., where two or more terms achieve the minimum.

Remark 2.8. The tropical hypersurface is a union of $(n-1)$-dimensional polyhedra (shapes defined by linear equalities and inequalities) each defined where two terms agree.

Definition 2.9. The *multiplicity of a tropical hypersurface* at such a polyhedron is the gcd of the entries of the difference of the exponent vectors on either side.

Example 2.10. The polynomial

$$x^2 \odot y \oplus x \odot y^2 \oplus x \oplus 1 \odot y$$

has the following multiplicity at one of the boundaries: The exponent vector of x is $(1, 0)$, and the exponent vector of $x \odot y$ is $(1, 2)$. The difference is $(0, -2)$, so the multiplicity is

$$\gcd(0, -2) = 2.$$

2.5. **Why multiplicities?** At each point v of a *plane curve* (a 1-dimensional tropical hypersurface) and each edge e containing v, there exists a unique vector u_e parallel to e, with integer entries with gcd $= 1$.

Theorem 2.11 (Balancing condition)**.** *At any point v of a plane curve, let E_v be the set of edges containing v. Then*

$$\sum_{e \in E_v} m_e u_e = 0,$$

where u_e is as above, and m_e is the multiplicity of e.

2.6. **Tropical lines.** Consider

$$a \odot x \oplus b \odot y \odot c,$$

where $a, b, c \in \mathbb{R}$. For two general points in the plane, there exists a unique tropical line passing through both of them.

Likewise, any two general tropical lines intersect at a unique point.

3. Igusa local zeta function, lecture 2

3.1. The Haar measure on $\mathbb{Z}_p$.

Recall:

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \right\} = \coprod_{a-0}^{p-1} (a + p\mathbb{Z}_p),$$

$$p\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p < 1 \right\},$$

$$\mathbb{Z}_p \setminus p\mathbb{Z}_p = \left\{ x \in \mathbb{Z}_p : |x|_p = 1 \right\}.$$

We have a basis of open sets of the form

$$a + p^n\mathbb{Z} = \left\{ x \in \mathbb{Z}_p : |x - a|_p \leq p^{-n} \right\},$$

where $a \in \mathbb{Z}_p$ and $n \in \mathbb{Z}^+$.

Let E be a union of sets of the form $a + p^n\mathbb{Z}_p$. Then the Haar measure on E has the following properties:

(1) $m(E) \geq 0$, $m(\varnothing) = 0$.
(2) If $E_1 \cap E_2 = \varnothing$, then

$$m(E_1 \cup E_2) = m(E_1) + m(E_2).$$

(Actually, we also have countable additivity.)
(3) $m(E) = m(a + E)$ for any $a \in \mathbb{Z}_p$.
(4) $m(\mathbb{Z}_p) = 1$.

So m is a countably additive, translation-invariant positive measure with total measure 1.

By translation invariance,

$$m(a + p^n\mathbb{Z}_p) = m(p^n\mathbb{Z}_p) = \frac{m(\mathbb{Z}_p)}{p^n} = \frac{1}{p^n}.$$

Also,

$$1 = \int_{\mathbb{Z}_p} dx = \int_{p\mathbb{Z}_p} dx + \int_{\mathbb{Z}_p \setminus p\mathbb{Z}_p} dx,$$

so

$$m(\mathbb{Z}_p \setminus p\mathbb{Z}_p) = \int_{\mathbb{Z}_p \setminus p\mathbb{Z}_p} dx = 1 - p^{-1}.$$

Likewise,

$$m(p^e\mathbb{Z}_p \setminus p^{e-1}\mathbb{Z}_p) = p^{-e} - p^{-(e+1)} = p^{-e}\left(1 - p^{-1}\right).$$

3.2. The Igusa local zeta function.

Let $f(x_1, x_2, \ldots, x_n) \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$, and let $s \in \mathbb{C}$ with $\operatorname{Re} s > 0$. Define

$$Z(s) = \int \cdots \int_{\mathbb{Z}_p^n} |f(x_1, x_2, \ldots, x_n)|^s \underbrace{dx_1\, dx_2 \ldots\, dx_n}_{\text{Haar measure on } \mathbb{Z}_p^n}.$$

Theorem 3.1 (Igusa, 1975). *$Z(s)$ is a rational function of $p^{-s} = T$ (using Hironaka's resolution of singularities, depending on p and $f(x)$).*

Example 3.2. Let $f(x) = x^n$. We have

$$\mathbb{Z}_p = \coprod_{e=0}^{\infty} p^e(\mathbb{Z}_p \setminus p\mathbb{Z}_p) \coprod \{0\}.$$

So

$$\begin{aligned}
\int_{\mathbb{Z}_p} |x|_p^{Ns}\,dx &= \sum_{e=0}^{\infty} \int_{p^e\mathbb{Z}_p \setminus p^{e+1}\mathbb{Z}_p} |x|_p^{Ns}\,dx \\
&= \sum_{e=0}^{\infty} \int_{p^e(\mathbb{Z}_p \setminus p\mathbb{Z}_p)} p^{-eNs}\,dx \\
&= \sum_{e=0}^{\infty} p^{-eNs} p^{-e}\left(1 - p^{-1}\right) \\
&= \left(1 - p^{-1}\right) \sum_{e=0}^{\infty} \left(p^{-Ns}p^{-1}\right)^e \\
&= \frac{1 - p^{-1}}{1 - p^{-Ns-1}} = \frac{1 - p^{-1}}{1 - p^{-1}T^N}.
\end{aligned}$$

Hence, we obtain

$$\int_{\mathbb{Z}_p} |x|_p^s\,dx = \frac{1 - p^{-1}}{1 - p^{-1}p^{-s}}.$$

An alternate method:

$$\begin{aligned}
Z(s) = \int_{\mathbb{Z}_p} |x|^{Ns}\,dx &= \int_{p\mathbb{Z}_p} |x|^{Ns} + \int_{\mathbb{Z}_p \setminus p\mathbb{Z}_p} |x|^{Ns}\,dx \\
&= \int_{\mathbb{Z}_p} |py|^{Ns} p^{-1}\,dy + \left(1 - p^{-1}\right) \\
&= p^{-Ns}p^{-1} \int_{\mathbb{Z}_p} |y|^{Ns}\,dy + \left(1 - p^{-1}\right) \\
&= p^{-Ns}p^{-1} Z(s) + \left(1 - p^{-1}\right),
\end{aligned}$$

thus

$$Z(s) = \frac{1 - p^{-1}}{1 - p^{-1}p^{-Ns}}.$$

Example 3.3.

$$\int_{\mathbb{Z}_p} \left|x^2(x-1)\right|_p^s\,dx = \int_{p\mathbb{Z}_p} \left|x^2\right|^s\,dx + \int_{1+\mathbb{Z}_p} |x-1|^s\,dx + (p-2)p^{-1}.$$

Example 3.4.

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} |x+y|^s\,dx\,dy = \sum_{f=0}^{\infty}\sum_{e=0}^{\infty} \int_{p^e(\mathbb{Z}_p \setminus p\mathbb{Z}_p)} \int_{p^f(\mathbb{Z}_p \setminus p\mathbb{Z}_p)} |x+y|^s\,dx\,dy = \frac{1 - p^{-1}}{1 - p^{-1}T}.$$

3.3. **Another form.** Recall that

$$Z(s) = \int_{\mathbb{Z}_p^n} |f(x_1, \ldots, x_n)|_p^s \, dx_1 \ldots dx_n.$$

So we can write

$$Z(s) = \sum_{e=0}^{\infty} m\big((x_1, \ldots, x_n) \bigm| f(x) = p^e u\big) \, T^e.$$

Observe that

$$Z(0) = m\big((x_1, \ldots, x_n) \bigm| f(x) = u\big)\,,$$
$$Z(1) = 1.$$

3.4. **Poincaré series.** Generating function:

$$|N_e| = \#\left\{(x_1, \ldots, x_n) \in (\mathbb{Z}/p^e\mathbb{Z})^n \bigm| f(x_1, \ldots, x_n) \equiv 0 \mod p^e\right\}$$
$$P(T) = \sum_{e=0}^{\infty} |N_e| \, p^{-ne} T^e.$$

Note that $|N_e| \leq p^{en}$ and $|N_0| = 1$.

Theorem 3.5 (Igusa, 1975)**.** *The Igusa zeta function can be expressed as*

$$Z(T) = P(T) - T^{-1}\big(P(T) - 1\big).$$

Equivalently,

$$P(T) = \frac{1 - Z(T)T}{1 - T}.$$

Proof. Observe that

$$\begin{aligned} Z(T) &= \sum_{e=0}^{\infty} m\big(x \in \mathbb{Z}_p^n \bigm| f(x) = p^e u\big) \, T^e \\ &= \sum_{e=0}^{\infty} \big(|N_e| \, p^{-en} T^e - |N_{e+1}| \, p^{-(e+1)n} T^e\big) \\ &= P(T) - T^{-1}\Big(\sum_{e=0}^{\infty} |N_{e+1}| \, p^{-(e+1)} T^{e+1}\Big) \\ &= P(T) - T^{-1}\big(P(T) - 1\big). \end{aligned}$$

□

4. Tropical geometry, lecture 2

4.1. Valuations.

Definition 4.1. A *valuation* on a field K is a function $v : K^* \longrightarrow \mathbb{R}$ such that:

1. $v(ab) = v(a) + v(b)$;
2. $v(a + b) \geq \min\{v(a), v(b)\}$.

Remark 4.2. By convention, $v(0) = \infty$.

Example 4.3. The p-adic valuation on $\mathbb{Q}$ or $\mathbb{Q}_p$.

Example 4.4 (Trivial valuation). For K any field, set $v(a) = 0$ for all $a \in K^*$.

Example 4.5 (Formal Laurent series). In the field $K((\pi))$ of formal Laurent series, define

$$v\Big(\sum_{i=-N}^{\infty} a_i \pi^i\Big) = \min\left\{i \mid a_i \neq 0\right\}.$$

Example 4.6 (Formal Puiseux series). The ring of *formal Puiseux series*

$$K\{\{\pi\}\} = \bigcup_{d\in\mathbb{Z}_{>0}} \mathbb{C}\big((\pi^{1/d})\big)$$

is algebraically closed if K is algebraically closed, and in characteristic zero, it is the algebraic closure of $K((\pi))$. We define a valuation

$$v\Big(\sum_{i=-N}^{\infty} a_i \pi^{i/d}\Big) = \min\left\{\frac{i}{d} \,\middle|\, a_i \neq 0\right\}.$$

Example 4.7. The algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ has an induced valuation.

Remark 4.8. For the rest of the talk, K will always be an algebraically closed field (of any characteristic) with valuation.

4.2. **Tropicalizing a Laurent polynomial in K.** Given

$$f = \sum_{i=1}^{m} a_i x_i^{e_{11}} \cdot \ldots \cdot x_n^{e_{1n}} \in K[x_1^{\pm}, \ldots, x_n^{\pm}],$$

define

$$\mathrm{Trop}(f) = \bigoplus_{i=1}^{m} v(a_i) \odot x_1^{e_{11}} \odot \ldots \odot x_n^{e_{1n}}.$$

Theorem 4.9 (Fundamental theorem of tropical geometry for hypersurfaces)**.** *If f is a Laurent polynomial in $K[x_1^{\pm}, \ldots, x_n^{\pm}]$, then*

$$V\big(\mathrm{Trop}(f)\big) \cap v(K^*) = \left\{\big(v(x_1), \ldots, v(x_n)\big) \,\middle|\, x_i \in K^*,\ f(x_1, \ldots, x_n) = 0\right\}.$$

Proof sketch. We show part of the proof of the "$\supseteq$" direction. Suppose $x_1, \ldots, x_n \in K^*$ such that $f(x_1, \ldots, x_n) = 0$. Note that

$$v(a_i x_1^{e_{i1}} \cdot \ldots \cdot x^{e_{in}}) = v(a_i) \odot v(x_1)^{e_{i1}} \odot \ldots \odot v(x_n)^{e_{in}}.$$

We want to show that the minimum of $\mathrm{Trop}(f)$ is achieved at least twice.

For contradiction, assume that the minimum is unique. Then $f(x_1, \ldots, x_n) \neq 0$ by the following lemma:

Lemma 4.10. *If $a, b \in K$ and $v(a) \neq v(b)$, then*

$$v(a+b) = \min\left\{v(a), v(b)\right\}.$$

The other direction ($\subseteq$) is hard! □

Example 4.11. If $K = \mathbb{C}\{\{\pi\}\}$ and $f = \pi x - y + 1$, then

$$\mathrm{Trop}(f) = 1 \odot x \oplus y \oplus 0.$$

We have

$$x = \pi^{-1/2} \qquad\qquad v(x) = -\frac{1}{2}$$

$$y = 1 + \pi^{1/2} v(y) = 0.$$

4.3. The fundamental theorem.

Theorem 4.12 (Fundamental theorem of tropical geometry)**.** *Let* $I \subset K[x_1^{\pm}, \ldots, x_n^{\pm}]$ *be an ideal. Then*

$$\operatorname{Trop}(I) \cap v(K^*) = \left\{ \big(v(x_1), \ldots, v(x_n)\big) \;\middle|\; (x_1, \ldots, x_n) \in V(I) \right\},$$

where

$$\operatorname{Trop}(I) = \bigcap_{f \in I} V(\operatorname{Trop} f),$$

$$V(I) = \left\{ (x_1, \ldots, x_n) \in (K^*)^n \;\middle|\; f(x_1, \ldots, x_n) = 0 \textit{ for all } f \in I \right\}.$$

Remark 4.13. The ideal I has a finite generating set, and for computing $V(I)$, it's sufficient to check the generators.

However, for computing $\operatorname{Trop}(I)$, it is *not* sufficient to take generators of I.

Definition 4.14 (Tropical bases)**.** If $f_1, \ldots, f_m$ are generators of I such that

$$\operatorname{Trop}(I) = \bigcap_{i=1}^{m} V(\operatorname{Trop} f_i),$$

then $f_1, \ldots, f_m$ is a *tropical basis*.

Remark 4.15. Tropical bases always exist (Gröbner bases).

Example 4.16. Let $K = \overline{\mathbb{Q}_3}$ and $n = 3$. Consider the ideal I generated by the tropical basis

$$f = xy + y - x + 3,$$
$$g = z^{-1} + 2 - 3x.$$

4.4. How to compute multiplicities.

Fix a point $p = (p_1, \ldots, p_n)$.

(1) Change coordinates so that the tropical variety is of the form $x_i = p$ for $d \le i \le n$, where $d = \dim I$.
(2) Choose "generic" a_i such that $v(a_i) = p_i$ for $i = 1, \ldots, d$.
(3) Count the points in

$$V(I) \cap V(x_1 = a_1, \ldots, x_d = a_d)$$

with $v(a_i) = p_i$ for $i = 1, \ldots, d$.

This is the multiplicity.

5. Igusa local zeta function, lecture 3

In this lecture, we will discuss Igusa's Stationary Phase Formula (1994) and Hironaka's Resolution of Singularities (1964) as methods of computing the Igusa local zeta function (ILZF).

Recall:

$$Z(s) = \int_{\mathbb{Z}_p^n} |f(x_1, \ldots, x_n)|^s \, dx_1 \ldots dx_n$$

is a *rational* function of $p^{-s} = T$.

5.1. **Poincaré series and ILZF.**

$$\begin{aligned} Z(T) &= P(T) - T^{-1}\left(P(T) - 1\right) \\ PT(T) &= \frac{1 - Z(T)T}{1 - T} \\ |N_e| &= \#\left\{(x_1, \ldots, x_n) \in (\mathbb{Z}/p^e\mathbb{Z})^n \;\middle|\; f(x_1, \ldots, x_n) \equiv 0 \mod p^e\right\}. \end{aligned}$$

Aside 5.1 (Special computation). Let $n = \#$ of variables of $f(x_1, \ldots, x_n)$. Recall from last time that

$$Z(T) = \sum_{e=0}^{\infty} \left[|N_e|\, p^{-ne} - |N_{e+1}|\, p^{-n(e+1)}\right] T^e.$$

Consider the special case

$$f(x_1, \ldots, x_n) = a + c_1x_1 + c_2x_2 + \cdots + c_nx_n + p(\text{some mess}),$$

where at least one $c_i \not\equiv 0 \mod p$. We have

$$|N_e| = p^{e(n-1)},$$

so

$$\begin{aligned} Z(T) &= \sum_{e=0}^{\infty} \left[p^{e(n-1)}p^{-en} - p^{(e+1)(n-1)}e^{-(e+1)n}\right] T^e \\ &= \sum_{e=0}^{\infty} \left[p^{-e} - p^{-(e+1)}\right] T^e \\ &= \sum_{e=0}^{\infty} \left(p^{-1}T\right)^e \left(1 - p^{-1}\right) \\ &= \frac{1 - p^{-1}}{1 - p^{-1}T}. \end{aligned}$$

5.2. **The stationary phase formula.**

Theorem 5.2 (Stationary phase formula, Igusa 1994)**.** *Let $f(x) \in \mathbb{Z}_p[x_1, \ldots, x_n]$, and write $T = p^{-s}$. Then*

$$Z(T) = \left(p^n - \left|\overline{N_1}\right|\right) p^{-n} + \left(\left|\overline{N_1}\right| - \left|\overline{S}\right|\right) p^{-n}T \left(\frac{1 - p^{-1}}{1 - p^{-1}T}\right) + \int_S |f(x)|^s \, dx,$$

where:

- $\overline{f(x)} \equiv f(x) \pmod p$,
- *$n = \#$ of variables in $\overline{f(x)}$,*
- *$\left|\overline{N_1}\right| = \#$ of vectors $x \in (\mathbb{Z}/p\mathbb{Z})^n$ such that $\overline{f(x)} \equiv 0 \pmod p$,*
- *$\left|\overline{S}\right| = \#$ of vectors $x \in \overline{N_1}$ such that $\frac{\partial \overline{f}}{\partial x_i}(x) \equiv 0 \pmod p$,*
- *$S =$ the set of all vectors $x \in (\mathbb{Z}_p)^n$ that are congruent mod p to vectors in $\overline{S}$.*

5.3. **Applications of SPF.**

Example 5.3. Let $f(x_1,x_2,x_3,x_4) = x_1x_2 + x_3x_4$. Then

$$\left|\overline{N_1}\right| = \left(p^2-1\right)p + p^2 = p^3+p^2-p$$

for $(x_1,x_3)\neq(0,0)$, and $\left|\overline{S}\right| = 1$, so

$$S = \mathbf{0} + (p\mathbb{Z}_p)^4.$$

Hence, we obtain

$$\begin{aligned} Z(T) = &\left(p^4-(p^3+p^2-p)\right)p^{-4} + \left(p^3-p^2-p-1\right)p^{-4}T\frac{1-p^{-1}}{1-p^{-1}T} \\ &+ \underbrace{\int_{(p\mathbb{Z}_p)^4} |x_1x_2+x_3x_4|^s \, dx_1\, dx_2\, dx_3\, dx_4}_{=I}. \end{aligned}$$

By a change of coordinates $x_i = py_i$, $dx_i = p^{-1}\,dy_i$,

$$\begin{aligned} I &= \int_{(\mathbb{Z}_p)^4} |py_1py_2+py_3py_4|^s \, p^{-4}\, dy_1\, dy_2\, dy_3\, dy_4 \\ &= p^{-4}T^2 \int_{(\mathbb{Z}_p)^4} |y_1y_2+y_3y_4|^s \, dy_1\, dy_2\, dy_3\, dy_4 \\ &= p^{-4}T^2 Z(T). \end{aligned}$$

So

$$\begin{aligned} Z(T) &= \left(1-p^{-1}\right)\left(1-p^{-2}\right) + \frac{(p^{-1}+p^{-2}-p^{-3}-p^{-4})T(1-p^{-1})}{1-p^{-1}T} + p^{-4}T^2Z(T) \\ Z(T) &= \frac{(1-p^{-1})(1-p^{-2})}{(1-p^{-1}T)(1-p^{-2}T)}. \end{aligned}$$

Example 5.4. Consider $f(x,y) = y^2-x^3$, $|N_1| = p$. Then

$$\begin{aligned} Z(T) &= \int_{\mathbb{Z}_p^2} \left|y^2-x^3\right|_p \, dx\, dy = \sum_{\xi\in(\mathbb{Z}/p\mathbb{Z})^2} \int_{\xi+(p\mathbb{Z}_p)^2} \left|y^2-x^3\right|_p^s \, dx\, dy \\ &= \left(p^2-p\right)p^{-2} + (p-1)p^{-2}T\frac{1-p^{-1}}{1-p^{-1}T} + \int_{(p\mathbb{Z}_p)^2} \left|y^2-x^3\right|_p^s \, dx\, dy. \end{aligned}$$

By applying the change of variables

$$\begin{aligned} x &= px_1 & dx &= p^{-1}\,dx_1 \\ y &= py_1 & dy &= p^{-1}\,dy_1, \end{aligned}$$

we obtain

$$Z(T) = \left(1-p^{-1}\right) + \frac{(1-p^{-1})p^{-1}T(1-p^{-1})}{1-p^{-1}T} + p^{-2}T^2\int_{(\mathbb{Z}_p)^2} \left|y_1^2-px_1^3\right|_p^s \, dx_1\, dy_1.$$

Denoting the integral at the end by I_1, write

$$f_1(x_1,y_1) = y_1^2 - px_1^3,$$

so that $\overline{f_1} = y_1^2 \equiv 0 \pmod{p}$, yielding

$$I_1 = \int_{\mathbb{Z}_p^2} \left|y_1^2 - px_1^3\right|_p^s \, dx_1 \, dy_1 = (p-1)p^{-1} + \int_{\mathbb{Z}_p \times p\mathbb{Z}_p} \left|y_1^2 - px_1^3\right|_p^s \, dx_1 \, dy_1.$$

Change variables again:

$$x_1 = x_2, \qquad dx_1 = dx_2,$$
$$y_1 = py_2, \qquad dy_1 = p^{-1} \, dy_2.$$

So

$$I_1 = \left(1 - p^{-1}\right) + p^{-1} T \int_{\mathbb{Z}_p^2} \left|py_2^2 - x_2^3\right|_p^s \, dx_2 \, dy_2.$$

Now we denote the remaining integral by I_2. We will need to apply SPF two more times.

Eventually, we obtain

$$Z(T) = \frac{(1-p^{-1})(1 - p^{-2}T + p^{-2}T^2 - p^{-5}T^5)}{(1-p^{-1}T)(1-p^{-5}T^6))},$$
$$Z(0) = \left(1 - p^{-1}\right) = \left(p^2 - p\right) p^{-2},$$
$$Z(1) = \frac{(1-p^{-1})(1-p^{-5})}{(1-p^{-1})(1-p^{-5})} = 1.$$

Remark 5.5. This method is equivalent to finding a resolution modulo p. This is an open question, so we don't know that it will work in general.

5.4. **Proof of SPF.** We have

$$Z(T) = \sum_{\xi \in (\mathbb{Z}/p\mathbb{Z})^n} \int_{\xi + p\mathbb{Z}_p^n} |f(x)|^s \, dx.$$

Let us consider the values of the integral for different ξ:

(1) If $\xi \in (\mathbb{Z}/p\mathbb{Z})^n \setminus \overline{N_1}$, then it is $(p^n - |N_1|)p^{-n}$.
(2) If $\xi \in \overline{N_1} \setminus \overline{S}$, then ...

6. Tropical geometry, lecture 3

6.1. **Abstract tropical curves.** Today's lecture is on the intrinsic viewpoint for tropical curves, i.e., how to get an abstract tropical curve from an embedded tropical curve.

(1) Take a 1-dimensional tropical curve with all edges having multiplicity 1.
(2) Label the "points at infinity" for the unbounded edges.
(3) Consider the tropical curve as a graph, including a vertex for each unbounded direction.
(4) Give the bounded edges lengths equal to their *lattice length*, the real number d such that

$$v - w = d(u_1, \ldots, u_n),$$

where $u_1, \ldots, u_n$ are integers with gcd 1.

Definition 6.1 (abstract tropical curve)**.** An *abstract tropical curve* is a finite connected graph G, together with *marked vertices* $v_1, \ldots, v_n$, all having degree 1, and positive lengths for each edge which doesn't contain a marked vertex. (The marked vertices correspond to unbounded edges.)

Definition 6.2 (genus)**.** The *genus* of a curve G is given by

$$g = E - V + 1 = \dim H_1(G, \mathbb{Q}),$$

where E is the number of edges, and V the number of vertices.

6.2. **Stable curves and stabilization.**

Definition 6.3. An abstract tropical curve is *stable* if every unmarked vertex has degree at least 3 and there exists at least one unmarked vertex.

There is a *stabilization* procedure to obtain a stable curve from an unstable curve:

(1) If an unmarked degree 1 vertex exists, delete it and the edge containing it.
(2) Repeat (1) as necessary.
(3) If any vertex has degree 2 (and is not part of a loop), replace the vertex and its edges with a single edge with length equal to the sum of the old lengths.
(4) Repeat (3) as necessary.

Remark 6.4. Stabilization doesn't change the genus.

Remark 6.5. Sometimes, we end up with a loop that has a vertex of degree 2 and get stuck!

Proposition 6.6. *A curve of genus g with n marked points has a stabilization if and only if $2g - 2 + n > 0$. If a stabilization exists, then it is unique.*

6.3. **Genus 0 stable curves: small cases.** What are the genus 0 stable curves with n marked points for $n \geq 3$?

Definition 6.7. The *combinatorial type* of a curve is the data of the curve without the lengths.

There is a unique stable curve of genus 0 with 3 marked points.

There are four combinatorial types of stable genus 0 curves with 4 marked points. Each of the first three cases has a length parameter $\ell \in \mathbb{R}_{>0}$. The fourth case (the "star tree") can be thought of as a limiting case $\ell = 0$ of the other three.

Therefore, the classification of stable curves of genus 0 with 4 marked points "is" three rays $\mathbb{R}_{\geq 0}$ glued along their vertex. In other words, the tropical curve $x \oplus y \oplus 0$ parametrizes stable genus 0 tropical curves with 4 marked points.

This is the *moduli space* of stable tropical curves of genus 0 with 4 marked points. A moduli space classifies not just as a set, but as a tropical variety.

6.4. **Genus 0 stable curves: the general case.** Embed the set of genus 0 stable curves with $n \geq 3$ marked points in $\mathbb{R}^N$:

(1) Pick arbitrary lengths for unbounded edges.
(2) Record, for each pair $1 \leq i < j \leq n$, the distance (sum of lengths of a path) from V_i and V_j. Let $d_{ij} = -$length,[2] giving a vector $(d_{ij}) \in \mathbb{R}^{\binom{n}{2}}$.
(3) Take the image of this vector in

$$\mathbb{R}^{\binom{n}{2}} / \mathbb{R}^n,$$

where $\mathbb{R}^n$ is the subspace of distances on the n-th star tree.

[2]The negative sign is so that the resulting moduli space will be a tropical variety.

Theorem 6.8. *The above defines an injective map*

$$\{\textit{genus } 0 \textit{ stable curves with } n \textit{ marked points}\} \hookrightarrow \mathbb{R}^{\binom{n}{2}-n}.$$

The image consists of the tropical variety

$$\bigcap_{i<j<k<\ell} V\left((d_{ij} \odot d_{k\ell} \oplus d_{ik} \odot d_{j\ell} \oplus d_{i\ell} \odot d_{jk}) \cdot d_{ij}^{-1} \odot d_{k\ell}^{-1}\right).$$

Remark 6.9. The terms in the above expression are known as the "tropical Plücker relations".

7. Tropical geometry, lecture 4

7.1. **Classical/tropical correspondences.** Classical geometry:

(1) 1 line through 2 points in general position
(2) 1 quadric through 5 general points
(3) 12 singular cubics through 8 general points

Tropical geometry:

(1) 1 line through 2 general points
(2) 1 quadric through 5 general points
(3) 12 singular cubics through 8 general points

In general, how many curves pass through k given general points? This is a hard question in classical geometry, and the answer wasn't known until the 1990s.

7.2. **The genus formula.** Classically, a smooth degree d curve has genus

$$g = \frac{(d-1)(d-2)}{2}.$$

A tropical reason for the genus formula: Arrange monomials of a general degree $d = 3$ curve in a triangle. Then

$$\begin{aligned}\text{genus of a tropical curve} &= \#\text{ holes of the tropical curve} \\ &= \#\text{ of points in triangle which are not on the edge} \\ &= \frac{(d-1)(d-2)}{2}.\end{aligned}$$

7.3. **Nodes.**

Definition 7.1. A *node* is a singular point where two smooth branches of the curve cross, i.e., the blow-up has two distinct points and is an immersion near each point.

Equivalently, there is a node at $(0,0)$ iff the equation for the curve is

$$xy + (\text{cubic and higher terms})$$

after a change of coordinates.

A degree d curve with n nodes has (geometric) genus equal to

$$\frac{(d-1)(d-2)}{2} - n.$$

Tropical analogue: When resolving a tropical node, introduce one new vertex, so

$$g = E - V + 1$$

decreases by one.

7.4. **Counting curves through general points.** Returning to the original question: How many irreducible curves of genus g and degree d pass through k given general points?

Proposition 7.2. *There are* $\left\{\begin{array}{c}\textit{infinitely many}\\ \textit{finitely many}\\ \textit{no}\end{array}\right\}$ *irreducible curves of degree d and genus g passing through k general points if*

$$k \left\{\begin{array}{c}<\\=\\>\end{array}\right\} g+3d-1.$$

Definition 7.3. $N_{g,d} = \#$ of irreducible curves of degree d and genus g through $k = g+3d-1$ general points.

Theorem 7.4 (Kontsevich 1994)**.**

$$N_{0,d} = \sum_{\substack{d_1+d_2=d\\ d_1,d_2>0}} \left[d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right] N_{0,d_1} N_{0,d_2}.$$

Theorem 7.5 (Mikhalkin 2005)**.** $N_{g,d}$ = *number of tropical curves through $g+3d-1$ general points (counted with multiplicity).*

Remark 7.6. Gathmann–Markwig reproved Kontsevich's formula in 2007/2008 using the above theorem of tropical geometry.

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