Singular Learning Theory Problems: Day 2

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22 May 2013

1. Given integers $\alpha_1, \ldots, \alpha_d \geq 0$, compute the RLCT at the origin of the following monomial ideals:
   (a) $\langle \omega_1^{\alpha_1}, \omega_2^{\alpha_2}, \ldots, \omega_d^{\alpha_d} \rangle$
   (b) $\langle \omega_1^{\alpha_1} \omega_2^{\alpha_2} \cdots \omega_d^{\alpha_d} \rangle$

2. Find the RLCT over all points $(t, \omega) \in \mathbb{R}^2$ of the following ideal in the ring $\mathbb{R}[t, \omega]$:
$$I = \left\langle \frac{1}{2}t + (1-t)\omega - \frac{1}{2}, \frac{1}{2}t + (1-t)(1-\omega) - \frac{1}{2} \right\rangle.$$

3. Given an integer $m \geq 0$, find the threshold (λ, θ) appearing in the following exponential integral
$$\int_{\mathbb{R}} e^{-nx^2} |x^m| dx \;\approx\; CN^{-\lambda} (\log N)^{\theta - 1}.$$
If C is known, can you prove that the above is not an approximation but an exact formula? Hint: Substitute $x = ty$, and study the limit as $t \to \infty$ of $Z(Nt^2)/(C(Nt)^{-\lambda}(\log Nt)^{\theta-1})$.

4. **Blowing up a singularity**. Find the RLCT of the ideal $\langle x^3 y - xy^3 \rangle$.

   Hint: You may need a transformation called a *blow up* to resolve the singularity at the origin. The blow up of the origin in $\mathbb{R}^2$ is a map $\rho : U \to \mathbb{R}^2$ where the manifold U is covered by two charts U_1, U_2. Each chart is isomorphic to $\mathbb{R}^2$ and the map ρ restricted to each chart is given by
$$\rho : U_1 \to \mathbb{R}^2, \quad (s, t) \mapsto (st, t),$$
$$\rho : U_2 \to \mathbb{R}^2, \quad (u, v) \mapsto (u, uv).$$

5. (Hard) **Sum and product rules for ideals with disjoint indeterminates**.
Given indeterminates $a_1, \ldots, a_s, b_1, \ldots, b_t$, suppose that the ideal I is generated by polynomials in $a_1, \ldots, a_s$ and the ideal J is generated by polynomials in $b_1, \ldots, b_t$. Let $\mathrm{RLCT}(I) = (\lambda_a, \theta_a)$ and $\mathrm{RLCT}(J) = (\lambda_b, \theta_b)$. Prove the following formulas for the sum and the product of I and J.
$$\mathrm{RLCT}(I + J) \;=\; (\lambda_a + \lambda_b,\; \theta_a + \theta_b - 1)$$
$$\mathrm{RLCT}(IJ) \;=\; \begin{cases} (\lambda_a,\; \theta_a) & \text{if } \lambda_a < \lambda_b, \\ (\lambda_b,\; \theta_b) & \text{if } \lambda_a > \lambda_b, \\ (\lambda_a,\; \theta_a + \theta_b) & \text{if } \lambda_a = \lambda_b. \end{cases}$$