Example to illustrate the connection between the Igusa Local Zeta function $Z(T)$ and its Poincaré series $P(T)$

We showed in class that

$$P(T) = \frac{1 - Z(T)T}{1 - T} = \frac{1}{1-t} - \frac{Z(T)T}{1-T}$$

From this relation follows that if $Z(T)$ is a rational function of $T = p^{-s}$ then so is $P(T)$.

Example Let $f(x_1, x_2, x_3, x_4) = x_1x_2 + x_3x_4$. Later using SPF we will show that

$$Z(T) = \frac{(1-p^{-1})(1-p^{-2})}{(1-p^{-1}T)(1-p^{-2}T)}.$$

We want to use $Z(T)$ and the relation above to find $|N_e|$ and $P(T)$. First we use partial fractions to find A and B such that

$$Z(T) = \frac{A}{1-p^{-1}T} + \frac{B}{1-p^{-2}T}$$

We find that $A = (1-p^{-2})$ and $B = -p^{-1}(1-p^{-2})$ so that

$$\begin{aligned}
Z(T) &= \frac{(1-p^{-2})}{1-p^{-1}T} - \frac{p^{-1}(1-p^{-2})}{1-p^{-2}T} \\
&= (1-p^{-2})[1 + p^{-1}T + p^{-2}T^2 + p^{-3}T^3 + \cdots] \\
&\quad -p^{-1}(1-p^{-2})[1 + p^{-2}T + p^{-4}T^2 + p^{-6}T^3 + \cdots] \\
&= (1-p^{-2})[1 - p^{-1} + (p^{-1} - p^{-3})T + (p^{-2} - p^{-5})T^2 \\
&\quad +(p^{-3} - p^{-7})T^3 + \cdots + (p^{-e} - p^{-2e-1})T^e + \cdots]
\end{aligned}$$

Now using the relation above

$$\begin{aligned}
P(T) &= \frac{1}{1-t} - \frac{Z(T)T}{1-T} \\
&= 1 + T + T^2 + T^3 + \cdots \\
&\quad -Z(T)T - Z(T)T^2 - Z(T)T^3 + \cdots \\
&= 1 + (1 - (1-p^{-1})(1-p^{-2}))T \\
&\quad +[1 - (1-p^{-2})p^{-1}(1-p^{-2}) - (1-p^{-1})(1-p^{-2})]T^2 \\
&\quad +(1 - (1-p^{-2})p^{-2}(1-p^{-3}) - (1-p^{-2})p^{-1}(1-p^{-2}) - (1-p^{-1})(1-p^{-2}))T^3 + \cdots \\
&= 1 + (p^3 + p^2 - p)p^{-4}T + (p^6 + p^5 - p^3)p^{-8}T^2 + \cdots \\
&\quad +(p^{3e} + p^{3e-1} - p^{2e-1})p^{-4e}T^e + \cdots
\end{aligned}$$

From the last line above we see that $|N_e| = p^{3e} + p^{3e-1} - p^{2e-1}$. Finally we sum the series above to computer the rational function for $P(T)$.

$$
\begin{aligned}
P(T) &= 1 + (p^3 + p^2 - p)p^{-4}T + (p^6 + p^5 - p^3)p^{-8}T^2 + \cdots \\
&\quad + (p^{3e} + p^{3e-1} - p^{2e-1})p^{-4e}T^e + \cdots \\
&= 1 + p^{-1}T + p^{-2}T - p^{-3}T \\
&\quad + p^{-2}T^2 + p^{-3}T^2 - p^{-5}T^2 \\
&\quad + p^{-3}T^3 + p^{-4}T^3 - p^{-7}T^3 + \cdots \\
&\quad + p^{-e}T^e + p^{-e-1}T^e - p^{-2e-1}T^e + \cdots
\end{aligned}
$$

Summing we get that

$$
\begin{aligned}
P(T) &= \frac{1}{1 - p^{-1}T} + \frac{p^{-2}T}{1 - p^{-1}T} - \frac{p^{-3}T}{1 - p^{-2}T} \\
&= \frac{1 - p^{-3}T}{(1 - p^{-1}T)(1 - p^{-2}T)}
\end{aligned}
$$