ON PARABOLIC CLOSURES IN COXETER GROUPS

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Abstract. For a finitely generated subgroup W' of a Coxeter system (W, S), there are finitely generated reflection subgroups $W_1, \dots, W_n$ of W, each containing W', such that any reflection subgroup of W containing W' contains one of the W_i as a standard parabolic subgroup. The canonical Coxeter generators of the W_i, and an expression for the parabolic closure of W' as a W-conjugate of a standard parabolic subgroup of W, may be effectively determined.

1. Introduction and statement of results

This note describes results on the structure of reflection subgroups of a Coxeter system (W, S) which together afford an algorithm for computing the parabolic closure of (i.e. the parabolic subgroup of minimal rank containing) a given finitely generated subgroup of W.

Recall that the standard parabolic subgroups of W are the subgroups $W_J = \langle J \rangle$ generated by subsets J of S, and the parabolic subgroups of W are the W-conjugates of the standard parabolic subgroups; these notions depend on S. Any reflection subgroup W' of W has a canonical set of Coxeter generators (depending on S) which we denote as $\chi(W')$. Notions of parabolic and standard parabolic subgroups, rank etc of W' are defined in terms of the Coxeter generators $\chi(W')$ of W'.

In Section 2, we provide more background on the above notions, and prove the following results.

Proposition 1. *Let W' be a finitely generated subgroup of W. Then there exist $n \in \mathbb{N}_{\geq 1}$ and finitely generated reflection subgroups $W_1, \dots, W_n$ of W such that each W_i contains W' and if W'' is a reflection subgroup of W with $W'' \supseteq W'$, then W'' contains W_i as a standard parabolic subgroup for some i with $1 \leq i \leq n$.*

Proposition 2. *Let W'' be a finitely generated reflection subgroup of W with $\chi(W'') = \{c_1, \dots, c_n\}$ where $n = |\chi(W'')|$. Choose a finite subset J of S such that $W'' \subseteq W_J$. Then W'' is a parabolic subgroup of W iff there exist pairwise distinct $s_1, \dots, s_n \in J$ and some $w \in W_J$ with $w(c_1 \cdots c_n)w^{-1} = s_1 \cdots s_n$, in which case $W'' = w^{-1} W_{\{s_1, \dots, s_n\}} w$.*

Corollary 3. *The parabolic closure of the finitely generated subgroup W' of W is the (unique) subgroup W_i in* Proposition 1 *which is of minimal rank among the subgroups W_j, for $j = 1, \dots, n$, which are parabolic in W.*

The proof of Proposition 1 shows that the sets of canonical Coxeter generators $\chi(W_i)$ can be effectively determined from a finite set of generators of W'. Proposition 2 provides an effective test for determining whether a finitely-generated reflection subgroup of W is parabolic, since it involves only finitely many tests for conjugacy of elements of the finitely-generated Coxeter group W_J, and the conjugacy problem for (finitely generated) Coxeter groups is solvable in general (see [5], [1]; in fact, it is known [10] that for any finitely generated Coxeter system (W, S), there is a computable constant $N \in \mathbb{N}$ such that if $x, y \in W$ are W-conjugate, there is an element $w \in W$ of length $l(w) \leq N(l(x) + l(y))$ with $y = wxw^{-1}$). An explicit expression of the parabolic closure of W' as a W-conjugate of a standard parabolic subgroup of W may therefore be effectively determined from a finite set of generators of W' using Corollary 3 and Proposition 2.

We remark that a quite different algorithm for computing the parabolic closure of a cyclic subgroup of W was given in [10], where it was used as a preliminary step in various polynomial time algorithms for solving the conjugacy problem. Despite Proposition 2, an effective test for conjugacy of finitely-generated reflection subgroups of W in general is not known to the author.

2. Background and proof of results

As general references for facts on Coxeter groups and their reflection representations, root systems, Bruhat order etc used here, see [2] and [9]

2.1. Let (W, S) be a Coxeter system, $l\colon W \to \mathbb{N}$ denote its standard length function, and $T = \{\, wsw^{-1} \mid w \in W, s \in S \,\}$ denote its set of reflections. For $w \in W$, let $N(w) := \{\, t \in T \mid l(tw) < l(w) \,\}$. Let $\leq$ denote Bruhat order on W, and $e = 1_W$.

A subgroup W' of W is called a reflection subgroup if it is generated by $W' \cap T$. Let W' be a reflection subgroup of W. Then by [7] or [6], W' has a canonical set of Coxeter generators $S' = \chi(W) := \{\, t \in T \mid N(t) \cap W' = \{t\} \,\}$. We say a subgroup of W' is a standard parabolic subgroup of W' if it is generated by a subset of $\chi(W')$. A subgroup of W' is called a parabolic subgroup of W' if it is conjugate in W' to a standard parabolic subgroup of W'. An algorithm for computing $\chi(W')$ from a finite set of reflections generating W' is described in [7] and in more detail in [6]. The cardinality of $\chi(W')$ will be called the rank of W'.

2.2. Here, we recall some general facts from [8]. Let $\Omega_{(W,S)}$ be the directed graph with vertex set W and with set of directed edges $\{\, (tw, w) \mid w \in W,\, t \in N(w) \,\}$, where we view (tw, w) with $t \in N(w)$ as an edge directed from tw to w. For $A \subseteq W$, let $\Omega_{(W,S)}(A)$ denote the full subgraph of $\Omega_{(W,S)}$ with vertex set A. Another characterization of $S' = \chi(W')$ amongst sets of Coxeter generators of W' is that for any $x \in W$, there is an element $y \in xW'$ such that the map $\Omega_{(W,S)}(xW') \to \Omega_{(W',S')}$ given by $w \mapsto y^{-1}w$ an isomorphism of directed graphs. The element y is the unique element of xW' with $N(y^{-1}) \cap W' = \emptyset$, the unique element of minimal length $l(y)$ in the coset xW', the unique source of the graph $\Omega_{(W,S)}(xW')$ and the unique element $y \in xW'$ such that there is no edge (yr, y) in $\Omega_{(W,S)}$ with $r \in \chi(W')$. It is easy to see from the last description of y that if W' is finitely generated (i.e. $\chi(W')$ is finite) then y is effectively computable from x and $\chi(W')$.

For use in the proof of Proposition 2, we record the following:

Lemma 1. *Let W'' be a reflection subgroup of W and $w \in W$. Fix $u \in wW''$ with $N(u^{-1}) \cap W'' = \emptyset$. Then $\chi(wW''w^{-1}) = u\chi(W'')u^{-1}$.*

Proof. Let $S'' := \chi(W'')$. Now $wW''w^{-1} = uW''u^{-1}$ clearly has $uS''u^{-1}$ as a set of generators. Since the set $\chi(uW''u^{-1})$ of Coxeter generators of $uW''u^{-1}$ is a minimal set of generators of $uW''u^{-1}$, it will suffice to show that $uS''u^{-1} \subseteq \chi(uW''u^{-1})$.

Let $s \in S''$ i.e. $s \in T$ with $N(s) \cap W'' = \{s\}$. To prove $usu^{-1} \in \chi(uW''u^{-1})$, we must show $N(usu^{-1}) \cap uW''u^{-1} = \{usu^{-1}\}$ or $u^{-1}N(usu^{-1})u \cap W'' = \{s\}$. To show this, regard N as a cocycle of W with values in the power set of T, regarded as additive abelian group under symmetric difference and with left W-action by conjugation, as in [7].

The cocycle condition gives $N(usu^{-1}) = N(u) + uN(s)u^{-1} + usN(u^{-1})su^{-1}$. Hence $u^{-1}N(usu^{-1})u = N(u^{-1}) + N(s) + sN(u^{-1})s$. Since $W'' = sW''s$, we get

$$\begin{aligned} u^{-1}N(usu^{-1})u \cap W'' &= (N(u^{-1}) \cap W'') + (N(s) \cap W'') + s(N(u^{-1}) \cap W'')s \\ &= \emptyset + \{s\} + \emptyset = \{s\} \end{aligned}$$

as required. □

2.3. **Proof of Proposition 1.** Suppose W' is generated by $x_1, \dots, x_m$. Let W'' be a reflection subgroup of W, and $S'' = \chi(W'')$. We have $x_i \in W''$ iff x_i has some reduced expression $x_i = r_n \cdots r_1$ in (W'', S'') iff there is a directed path $e = w_0, w_1, \dots, w_n = x_i$ in $\Omega_{(W'',S'')}$ with $w_j w_{j-1}^{-1} = r_j \in S''$ for $j = 1, \dots, n$ iff there is a directed path $e = w_0, w_1, \dots, w_n = x_i$ in $\Omega_{(W,S)}$ with $w_j w_{j-1}^{-1} = r_j \in S''$ for $j = 1, \dots, n$. Note that there are only finitely many directed paths from e to x_i in $\Omega_{(W,S)}$, since all vertices y of such a path are in the (finite) Bruhat interval $[e, x_i]$ and the standard length function of (W, S) strictly increases along the path.

Consider the subsets R of T such that $R = \chi(\langle R \rangle)$ and for each $i = 1, \dots, m$, there is some directed path $e = w_0, w_1, \dots, w_n = x_i$ (with n and the w_j depending on i) in $\Omega_{(W,S)}$ with $w_j w_{j-1}^{-1} \in R$ for $j = 1, \dots, n$. From above, such sets R are precisely the sets of canonical Coxeter generators of the reflection subgroups of (W, S) which contain W'. The above remarks also imply that the set of such subsets R, when ordered by inclusion, has only finitely many minimal elements, say $R_1, \dots, R_n$, and that the reflection subgroups $W_1, \dots, W_n$ defined by $W_i = \langle R_i \rangle$ for $i = 1, \dots, n$ have the required properties. In fact, any minimal set R as above is a subset of the finite set $T \cap \{\, xy^{-1} \mid x \le y \le x_i \text{ for some } i = 1, \dots, n \,\}$, so the R_i are effectively computable.

2.4. Recall that a parabolic subgroup of a parabolic subgroup of W is itself a parabolic subgroup of W. Further, the intersection of two parabolic subgroups of W is a parabolic subgroup of both of them, and hence also a parabolic subgroup of W, by a well-known result of Kilmoyer (see [4, Theorem 2.7.4]; the proof there for the case of finite W readily extends to arbitrary W). It follows that the intersection W'' of all parabolic subgroups containing a given finitely generated subgroup W' of W is the unique parabolic subgroup W'' of W of minimal (finite) rank containing W'. One calls W'' the parabolic closure of W'. Note that W'' is a parabolic subgroup of any (finite rank) standard parabolic subgroup of W which contains W'.

2.5. **Proof of Proposition 2.** Suppose first that W'' is a parabolic subgroup of W. The remarks of the preceding subsection show that W'' is a parabolic subgroup of W_J, so $wW''w^{-1} = W_K$ for some $K \subseteq J$ and $w \in W_J$. Write $w = yz$ where $z \in W''$ and $N(y^{-1}) \cap W'' = \emptyset$. Then $W_K = yW'y^{-1}$ so by Lemma 1, $K = \chi(W_K) = y\chi(W')y^{-1} = \{\, yc_1y^{-1}, \dots, yc_ny^{-1} \,\}$. Letting $yc_iy^{-1} = s_i \in K$, we have $y(c_1 \cdots c_n)y^{-1} = s_1 \cdots s_n$ as required.

Conversely, suppose $s_1, \dots, s_n$ in J are pairwise distinct and $w(c_1 \cdots c_n)w^{-1} = s_1 \cdots s_n = c$. We may assume without loss of generality that W is realized in its standard reflection representation (see e.g. [9, Ch 5]) on a real vector space V with standard symmetric bilinear form, with associated root system Φ and linearly independent simple roots Π. For $\alpha \in \Phi$, we let $s_\alpha \in W$ denote the orthogonal reflection in α. By enlarging V and extending the form if necessary, we may assume that the form $(?|?)$ on V is non-degenerate.

Let $s_i = s_{\alpha_i}$ and $c_i = s_{\beta_i}$ where $\alpha_i \in \Pi$ and $\beta_i \in \Phi$. Using non-degeneracy of the form, it is easy to see that $\operatorname{Im}(e-c) = V' := \sum_{i=1}^n \mathbb{R}\alpha_i$ is n-dimensional. On the other hand, if $\gamma_i, \ldots, \gamma_m \in \Phi$ with $c = s_{\gamma_1}\cdots s_{\gamma_m}$, then $\operatorname{Im}(e - s_{\gamma_1}\cdots s_{\gamma_m}) \subseteq \sum_{i=1}^n \mathbb{R}\gamma_i$ is at most m-dimensional. Hence if $c = s_{\gamma_1}\cdots s_{\gamma_m}$ with $m \leq n$ we get that $m = n$ and $\gamma_1, \ldots, \gamma_n$ is a $\mathbb{R}$-basis of V'. It is well known (see e.g. [3, Proposition 3.3]) that $\gamma_i \in V'$ implies $s_{\gamma_i} \in W_K$ for $i = 1, \ldots, n$, where $K = \{s_1, \ldots, s_n\}$.

Applying the preceding paragraph with $\gamma_i = w(\beta_i)$ shows that $\gamma_i \in V'$ and $s_{\gamma_i} = wc_iw^{-1} \in W_K$ for $i = 1, \ldots, n$. Write $w = yz$ where $z \in W''$ and $N(y^{-1}) \cap W'' = \emptyset$. Since $W'' = \langle\, zc_iz^{-1} \mid i = 1, \ldots, n\,\rangle$, it follows that $yW''y^{-1} \subseteq W_K$. Choose a reduced expression $p = r_1 \cdots r_k$ for $p := zc_1\cdots c_nz^{-1}$ in $(W'', \chi(W''))$. Then there is a directed path $e, r_k, r_{k-1}r_k, \ldots, r_1\cdots r_{k-1}r_k = p$ in $\Omega_{(W'',\chi(W''))}$. By Lemma 1, conjugating by y gives a directed path

$$e, yr_ky^{-1}, yr_{k-1}r_ky^{-1}, \ldots, yr_1\cdots r_{k-1}r_ky^{-1} = ypy^{-1} = s_1\cdots s_n$$

in $\Omega_{(yW''y^{-1},\chi(yW''y^{-1}))}$. The last path is also a directed path in $\Omega_{(W,S)}$. Now since the s_i are pairwise distinct, any path $e = q_0, q_1, \ldots, q_l = s_1\cdots s_n$ in $\Omega_{(W,S)}$ has $l = n$ and satisfies $\langle\, q_iq_{i-1}^{-1} \mid i = 1, \ldots, n\,\rangle = \langle\, s_1, \ldots, s_n\,\rangle$ (for example, by induction on n). Hence $k = n$ and $\langle\, yr_iy^{-1} \mid i = 1, \ldots, n\,\rangle = W_K$. Thus, $yW''y^{-1} \supseteq W_K$. Since the reverse inclusion has previously been established and $z \in W''$, we get $wW''w^{-1} = yW''y^{-1} = W_K$, completing the proof.

2.6. **Proof of Corollary 3.** Corollary 3 follows immediately from the remarks in 2.4 since they imply that the parabolic closure of W' must be one of the subgroups W_i in Proposition 1.

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