Review Sheet for Exam 1

Standard disclaimer: The following represents a sincere effort to help you prepare for our exam. It is not guaranteed to be perfect. There might well be minor errors or (especially) omissions. These will not, however, absolve you of the responsibility to be fully prepared for the exam. If you suspect a problem with this review sheet, please bring it to my attention.

Format: the exam will consist of two parts. On Tuesday October 11 from 6:20-8:20 PM in Hayes-Healy 125, we'll have a closed book, closed note portion. It should take less than the full two hours to complete. The last page of the exam will contain a few proof-oriented take home problems whose solutions will be due (in my office or mailbox) by noon on Friday Oct 14. You should work on the take home problems by yourself, not consulting with other students, the web, etc. However, you're welcome to refer to your notes and the textbook, and you can talk to me if you have any questions.

Problems on the closed book portion will fall into three general categories.

- Statements of definitions and important theorems.
- Binary 'true/false' type problems. You will be required to provide *specific* counterexamples to false items and perhaps answer other questions about true items.
- 'Computational' problems.

Below I try to indicate more specifically what I want you to know in each category.

Things to know:

definitions and statements.: You should be able to give valid definitions of *linear combination*, *subspace*, *span of a set*, *(in)dependent set*, *generating set*, *basis*, *dimension*, *complementary subspace*, *linear transformation*, *nullspace*, *kernel*, *range*, *column space*, *rank*, *injective/surjective/bijective function*, *invertible function*, *invertible matrix*, *vector space isomorphism*, *similar matrices*, *row equivalent matrices*, *coordinates* of a vector. The only theorem I expect you to be able to state is the *rank theorem*.

knowledge useful for answering true/false questions.: Recognizing when a subset of a vector space is a subspace or a basis for a subspace. Understanding linear independence. Recognizing when a function between two vector spaces is a linear transformation. Recognizing injective/surjective/bijective functions. Identifying the dimension of a vector space. Identifying the kernel and range and of a linear transformation. Understanding the relationship between linear transformations and matrices.

computational skills.: Using row reduction (aka Gaussian elimination) to solve linear systems and interpreting the results of row reduction for answering various kinds of questions. Establishing linear independence of a set of vectors; finding a basis for and computing the dimension of a vector space (particularly the nullspace and range of a linear transformation). Finding coordinates (i.e. components) of a vector relative to a given basis (and vice versa). Finding and using the matrix for a linear transformation relative to given bases. Finding change of coordinate matrices. Describing an isomorphism between two given vector spaces. Addition, multiplication, inversion of matrices.

Things that won't be on the exam: I won't ask you to define *vector space* or *field*. I will not refer to theorems in the book by number or expect you to refer to them that way. I will avoid asking questions concerning inner products except where they come up in describing certain kinds of linear transformations (e.g. orthogonal projections, reflections, etc). The only field that will matter (i.e. the only one you will need for computations) on the exam is $\mathbf{R}$.

Other resources: Please keep in mind that in addition to your lecture notes and textbook, there are solutions to each homework assignment. There are some other linear algebra textbooks on reserve in the math library. Beyond the other problems in our textbook, I recommend Bretscher's book as a source for lots of nice, mostly straightforward problems.

Editorial comment (many ways of saying the same thing): One thing that is crucial to understanding what's going on this class is to remember that we now have several equivalent ways to express a linear system of equations. To illustrate this, I let A be an $m \times n$ matrix with jth column $\mathbf{a}_j \in \mathbf{R}^m$ and ij entry $a_{ij} \in \mathbf{R}$, and we let $\mathbf{x} \in \mathbf{R}^n$, $\mathbf{b} \in \mathbf{R}^m$ be vectors with ith entries x_i, b_i respectively. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$. Then the following equations all say exactly the same thing.

m equations in n unknowns.:

$$\begin{array}{rcrcl} a_{11}x_1 + a_{12}x_2 + & \ldots & a_{1n}x_n & = & b_1. \\ a_{21}x_2 + a_{22}x_2 + & \ldots & a_{2n}x_n & = & b_2. \\ \vdots & & & & \vdots \\ a_{m1}x_2 + a_{m2}x_2 + & \ldots & a_{mn}x_n & = & b_m. \end{array}$$

vector equation:

$$x_1\mathbf{a_1} + x_2\mathbf{a_2} + \cdots + x_n\mathbf{a_n} = \mathbf{b}.$$

matrix/vector equation:

$$A\mathbf{x} = \mathbf{b}.$$

linear transformation equation:

$$T(\mathbf{x}) = \mathbf{b}.$$

A question about any of these equations will likely reduce to some question about what happens when we row reduce the augmented matrix

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & & & & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{pmatrix}$$

to put it in echelon or reduced echelon form. For instance, the questions "is $\mathbf{b}$ in the range of T?" and "does $\mathbf{b}$ belong to span $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$?" are the same as determining whether the augmented matrix represents a consistent system: i.e. "if we reduce $[A|\mathbf{b}]$ to echelon form, do all pivots occur before the last column?" Many statements and computations in linear algebra that seem mysterious at first are really just straightforward consequences of thinking about a linear system from as many of the above points of view as possible.