Existence and uniqueness theorem for nth order linear ODEs.

The general nth order linear ODE if there is given by

(1) $$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t) = b(t),$$

where $a_j : I \to \mathbf{R}$, $b : I \to \mathbf{R}$ are functions defined on some open interval $I \subset \mathbf{R}$. The ODE becomes an initial value problem if we further impose that

(2) $$y(t_0) = y_0, \ldots, \frac{d^{n-1}y}{dt^{n-1}}(t_) = y_{n-1}$$

for some $t_0 \in I$ and $y_0, \ldots, y_{n-1} \in \mathbf{R}$. The fundamental theorem concerning solutions of such problems is as follows.

Theorem 0.1. *Suppose that the coefficient functions $a_j : I \to \mathbf{R}$ and the right side $b(t)$ in 1 are all continuous on I. Then there is a unique n-times differentiable function $y : I \to \mathbf{R}$ satisfying both 1 and 2.*

Linearity of the ODE simplifies the statement of the theorem in two ways. First of all, the hypothesis on the ODE becomes a hypothesis on the coefficients a_j which is more intuitive than in the general case (even general first order case). Secondly, and more importantly, there is no question about the domain of the solution. It is exactly the same as the domain of the functions a_j and b that define the ODE.

The uniqueness part of the theorem should be particularly useful in helping you deal with problem 2 on homework 5 (in somewhat the same way I used uniqueness to analyze solutions of $y' = y$ in class).