Upper Triangular Matrices and Normal Operators

In class I got hung up on establishing the following fact, which Treil uses in his proof of the spectral theorem for normal operators.

Proposition 0.1. *Let* $A \in M_{n\times n}(\mathbf{C})$ *be an upper triangular matrix such that* $\bar{A}^T A = A\bar{A}^T$. *Then* A *is diagonal.*

Treil proves this fact himself in the text. He also observes (earlier) that the corresponding fact in the self-adjoint case ('If $A = \bar{A}^T$ is upper triangular, then A is diagonal') is obvious. Nevertheless, in order to assuage my guilty conscience, I give you my argument, cleaned up and corrected, in writing.

Proof. Let the columns of A be denoted in order by $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and the rows of A by $\mathbf{w}_1, \ldots, \mathbf{w}_n$. Then with a little effort, one sees that the ij-entry of $\bar{A}^T A$ is $\overline{\mathbf{v}_i \cdot \mathbf{v}_j} = \{\mathbf{v}_j \cdot \mathbf{v}_i\}$ (I think I forgot the bar in class). In particular, the ii-element of $\bar{A}^T A$ is $\|\mathbf{v}_i\|^2$. Similarly, the ii-entry of $A\bar{A}^T$ is $|\mathbf{w}_i|^2$. So if A and $\bar{A}^T$ commute, we must have that the norm of the ith row of A is the same as the norm of the ith column.

Now suppose, in order to reach a contradiction, that A is not diagonal. Let i be the smallest index such that the ith row of A contains a non-zero off-diagonal element a_{iJ}, for some $i < J \leq n$. Then

$$\|\mathbf{w}_i\|^2 = \sum_{j=1}^{n} |a_{ij}|^2 \geq |a_{ii}|^2 + |a_{iJ}|^2.$$

But on the other hand, the ith column of A contains no non-zero off diagonal elements. We have $a_{ji} = 0$ for $j > i$ because A is upper triangular, and $a_{ji} = 0$ for $j < i$, by our choice of i. Hence

$$\|\mathbf{v}_i\|^2 = |a_{ii}|^2 < |a_{ii}|^2 + |a_{ij}|^2 \leq \|\mathbf{w}_i\|^2,$$

which by the first paragraph is inconsistent with the fact that A and $\bar{A}^T$ commute. I conclude that there are no non-zero off-diagonal elements in A. □