Assignment 1
Math 60380, Winter '10 Due Friday, February 5

Textbook problems:

GK, p249: 65.

Solution. False. For a counterexample choose any negative subharmonic function that is not harmonic. For instance, $u(z) = \log|z|$ on $D(0,1)$. Or (if you prefer something bounded) $u(z) = |z|^2 - 1$ on $D(0,1)$.

Problem 1. (This problem expands on exercise 5 in Krantz; it also depends on your knowing a bit about differential forms—wedge product and Green's/Stokes' Theorem mostly). Let $\eta = A\,dx + B\,dy$ be a 1-form on an open set $\Omega \subset \mathbf{C}$. We define $*\eta$ to be the 1-form $-B\,dx + A\,dy$.

(a) Let $u : \Omega \to \mathbf{R}$ be a C^2 function. One calls $*du$ (sometimes written '$d^c u$') the *conjugate differential* of u. Show that $d * du = \Delta u\,dx \wedge dy$. Hence $*du$ is closed if and only if f is harmonic.

Solution. I compute

$$\begin{aligned} d * du &= d(-u_y\,dx + u_x\,dy) = (-u_{xy}\,dx + -u_{yy}\,dy) \wedge dx + (u_{xx}\,dx + u_{yx}\,dy) \wedge dy \\ &= -u_{yy}\,dy \wedge dx + u_{xx}\,dx \wedge dy = (u_{xx} + u_{yy})\,dx \wedge dy = \Delta u\,dx \wedge dy. \end{aligned}$$

The third equality holds because $dx \wedge dx = dy \wedge dy = 0$ and the fourth because $dy \wedge dx = -dx \wedge dy$.

(b) Show that if v is a second C^2 function, then $du \wedge *dv = dv \wedge *du$.

Solution.

$$du \wedge *dv = (u_x\,dx + u_y\,dy) \wedge (-v_y\,dx + v_x\,dy) = u_x v_x\,dx \wedge dy - u_y v_y\,dy \wedge dx = (u_x v_x + u_y v_y)\,dx \wedge dy.$$

Since the last expression is symmetric in u and v, it follows that it is also equal to $dv \wedge *du$.

(c) Show that if u is harmonic and Ω is simply connected that $*du = dv$ where $v : \Omega \to \mathbf{R}$ is any harmonic conjugate for u. Deduce from this a simple expression for $*d \log|z|$.

Solution. Since Ω is simply connected, we know there exists a harmonic conjugate v for u. Since $u+iv$ is holomorphic, it follows from the Cauchy-Riemann equations that

$$dv = v_x\,dx + v_y\,dy = -u_y\,dx + u_x\,dy = *du.$$

Now v is a harmonic conjugate for $\log|z|$ in some domain if and only if $v(z) = \theta + C$ where θ is the argument of z and C is a complex constant. Hence $*d\log|z| = dv = d\theta$.

(d) Let γ be a C^1 curve in Ω. Show that in more classical language, one has $\int_\gamma *du = \int_\gamma \frac{\partial u}{\partial n} |d\gamma|$ where n is the righthand normal vector to γ.

Solution. The unit tangent vector to γ is given by $(\gamma_1', \gamma_2')/|\gamma'|$. The right hand normal n to γ is therefore obtained by rotating this vector $\pi/2$

radians clockwise. Thus $n = (\gamma_2', -\gamma_1')/|\gamma'|$ and $\frac{\partial u}{\partial n} = \nabla u \cdot n = \frac{-u_x\gamma_2' + u_y\gamma_1'}{|\gamma'|}$. From this, I infer

$$\int_\gamma *du = \int_\gamma -u_y\,dx + u_x\,dy = \int (-u_y\gamma_1' + u_x\gamma_2')\,dt = \int \frac{\partial u}{\partial n}|\gamma'(t)|\,dt = \int_\gamma \frac{\partial u}{\partial n}|d\gamma|.$$

(e) Show that if $\Omega' \subset \Omega$ is a bounded open subset with smooth boundary $b\Omega' \subset \Omega$, and if $u, v : \Omega \to \mathbf{R}$ are C^2 functions, then

$$\int_{b\Omega'} u * dv - v * du = \iint_{\Omega'} (u\Delta v - v\Delta u)\,dx\,dy.$$

Solution. Green's/Stokes' Theorem gives me that

$$\int_{b\Omega'} u * dv - v * du = \iint_{\Omega'} d(u * dv - v * du) = \iint_{\Omega'} du \wedge *dv + u\,d*dv - dv \wedge *du + v\,d*du.$$

From the first two parts of this problem, I see that the last integral is the same as $\iint_{\Omega'}(u\Delta v - v\Delta u)\,dx\,dy$.

Problem to be continued on next assignment...

Problem 2. Let $\Omega \subset \mathbf{R}^2 = \mathbf{C}$ be open. As with functions on the real line, one calls a function $\psi : \Omega \to \mathbf{R}$ of *two* real variables convex if $\psi(\frac{z+w}{2}) \leq \frac{1}{2}(\psi(z) + \psi(w))$ for all $z, w \in \mathbf{R}^2$. One can show (and you can take for granted) that convex functions are automatically continuous. Given this, show that a convex function is subharmonic. Show by example that a subharmonic function need not be convex.

Solution. If ψ is convex and $\overline{D(P,R)} \subset \mathbf{C}$, then for any $\theta \in \mathbf{R}$, we have

$$\psi(P) \leq \frac{1}{2}(\psi(P + Re^{i\theta}) + \psi(P - Re^{i\theta})) = \frac{1}{2}(\psi(P + Re^{i\theta}) + \psi(P + Re^{i(\pi+\theta)}).$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} \psi(P+Re^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{\pi} (\psi(P+Re^{i\theta})+\psi(P+Re^{i(\pi+\theta)}))\,d\theta \geq \frac{1}{\pi}\int_0^{\pi} \psi(P)\,d\theta = \psi(P).$$

So ψ satisfies the subaveraging property and is therefore subharmonic. □

To see that a subharmonic function need not be convex, consider $\log|z|$ which is subharmonic on $\mathbf{C}$. However, the restriction of this function to the positive real axis is $\log x$, which is actually strictly concave down everywhere. So $\log(\frac{x+y}{2}) > \frac{\log x + \log y}{2}$ for all $x, y > 0$.

Problem 3. Let $\Omega = (a,b) \times \mathbf{R} \subset \mathbf{C}$ be an open vertical strip and $u : \Omega \to [-\infty, \infty)$ be given by $u(x,y) = \psi(x)$ (i.e. u is really a function of only one variable). Show that u is subharmonic if and only if ψ is convex. (Hint: show that if u is not convex, then after subtracting the right harmonic function from u, the difference violates the maximum principle).

Solution. If ψ is convex, then so is u. Hence, from the previous problem, it follows that u is subharmonic.

Suppose, on the other hand, that ψ is not convex. Then there exist real numbers $a < b < c$ such that $\psi(b) > \ell(b)$, where $\ell : \mathbf{R} \to \mathbf{R}$ is the affine function agreeing with ψ at a and c. Since $\psi - \ell$ is continuous, we may choose $x_0 \in (a, c)$ such that $\psi(x_0) - \ell(x_0) > 0$ is maximal. Note that $h(x, y) = \ell(x)$ is a harmonic function on $\mathbf{C}$. So u is subharmonic if and only if $u - h$ is. On the other hand, $u - h$ is a non-constant function (it's equal to zero at any point $a + iy$ but positive at any point $x_0 + iy$) with an interior local maximum at any point of the form $x_0 + iy$. That is, $u - h$ does not satisfy the maximum principle and is therefore not subharmonic. □

Problem 4. ('Radial' subharmonic functions) Let $\Omega = \{R_1 < |z| < R_2\}$ be an annulus and $u : \Omega \to [-\infty, \infty)$ be given by $u(re^{i\theta}) = f(r)$ for all points $re^{i\theta} \in \Omega$ (i.e. u is a 'radial' function, with $u(z)$ depending only on the distance of z from 0). Show that u is subharmonic if and only if f is a convex function of $\log r$ (i.e. $f(e^x)$ is a convex function of $x = \log r$). (Hint: reduce to the previous problem). From this, give an explicit description (i.e. a formula) for any radial *harmonic* function on Ω.

Solution. Note that the function $f(z) = e^z$ maps the open vertical strip $\Omega' = (\log R_1, \log R_2) \times \mathbf{R}$ onto Ω. While f is not $(1, 1)$, we see that $f'(z) = e^z$ never vanishes. Hence f is at least locally invertible. Since subharmonicity is a local property, it follows that u is subharmonic on Ω if and only if $u \circ f$ is subharmonic on Ω'. Also, since $u(re^{i\theta}) = f(r)$ is a radial function, we have that $u \circ f(z) = u(e^x e^{iy}) = f(e^x)$ is a function of $x = \operatorname{Re} z$ only. Hence by the previous problem, u is subharmonic if and only if $f(e^x)$ is a convex function of x. □

A radial function $h(re^{i\theta}) = f(r)$ is *harmonic* if and only if h and $-h$ are both subharmonic. By the first part of the problem, this is true if and only if $f(e^x)$ and $-f(e^x)$ are both convex, which is to say that f is an affine function of $x = \log r$. So h is harmonic if and only if there exist real constants α, β such that $h(re^{i\theta}) = \alpha \log r + \beta$ for all $re^{i\theta} \in \Omega$.

Problem 5. Show that if $f : \Omega' \to \Omega$ is holomorphic and $u : \Omega \to [-\infty, \infty)$ is subharmonic, then $u \circ f$ is subharmonic. (Hint: since we showed in class that this is true when f is injective, and since being subharmonic is a local property, it more or less suffices to establish the subaveraging property about points P at which $f'(P) = 0$. For the latter, it might help to use problem 1 on homework 9 from last semester.)

Solution. If f is constant, the assertion is clear, so suppose that f is not constant.

As noted in class, it suffices to show for each $P \in \Omega'$ that $u \circ f$ is subharmonic on a neighborhood of P, now if $f'(P) \neq 0$, it follows that there is a neighborhood $V \ni P$ such that $f : V \to f(V) \subset \Omega'$ is invertible. Hence, as we showed in class $u \circ f$ is subharmonic on V. If $f'(P) = 0$, on the other hand, then we showed in homework last semester that there exists a neighborhood $V \ni P$, $R > 0$, and

$k \geq 2$ such that $f = g^k$ where g maps V conformally onto $D(0, R)$. Hence $u \circ f$ is subharmonic on V if and only if $u(w^k)$ is subharmonic on $D(0, R)$. To see that $u(w^k)$ is subharmonic, fix any $r > 0$ smaller than R. Then

$$\int_0^{2\pi} (u(re^{i\theta})^k)\, d\theta = \int_0^{2\pi} u(re^{ik\theta})\, d\theta = \int_0^{2\pi ik} u(e^{i\phi})\, \frac{d\phi}{k} = \int_0^{2\pi} u(e^{i\phi})\, d\phi \geq u(0).$$

That is, $u(w^k)$ has the subaveraging property for small enough disks centered at 0. It follows that $u(w^k)$ is subharmonic on $D(0, R)$ and therefore that $u \circ f$ is subharmonic everywhere on Ω'. □