Assignment 2
Math 60380, Winter '10
Due Friday, February 19

Problem 1. Let $U \subset \mathbf{C}$ be open and $u : U \to \mathbf{R}$ be a C^2 function. Suppose that $\overline{D(P,R)} \subset U$.

(a) Apply the last part of the first problem on homework 1 with domain $\Omega' = \Omega_\epsilon$ equal to the annulus $\{\epsilon < |z - P| < R\}$, and then let $\epsilon \to 0$ to prove that

$$2\pi u(P) = \int_0^{2\pi} u(P + Re^{i\theta})\, d\theta + \iint_{D(P,R)} \Delta u(z) \log \frac{|z-P|}{R}\, dx\, dy.$$

Solution. Replacing z with $z - P$ throughout, it suffices to consider the case $P = 0$. Let $v(z) = \log \frac{|z|}{R} = \log|z| - \log R$. Then as observed on last week's homework, we have $*dv = d\theta$. Since v is harmonic except at $z = 0$ and vanishes on $bD(0,R)$, the formula

$$\int_{b\Omega'} u * dv - v * du = \iint_{\Omega'} (u\Delta v - v\Delta u)\, dx\, dy$$

becomes

$$\int_0^{2\pi} u(Re^{i\theta})\, d\theta - \int_0^{2\pi} u(\epsilon e^{i\theta})\, d\theta + \int_{bD(0,\epsilon)} v * du = -\iint_{\Omega_\epsilon} \Delta u(z) \log \frac{|z|}{R}\, dx\, dy.$$

Adding and subtracting $2\pi u(0)$ in the second integral, using $v(z) = \log(\epsilon/R)$ in the third integral and rearranging gives

$$\begin{aligned} 2\pi u(0) &= \int_0^{2\pi} u(Re^{i\theta})\, d\theta - \int_0^{2\pi} (u(\epsilon e^{i\theta}) - u(0))\, d\theta - \log\frac{\epsilon}{R} \int_{bD(0,\epsilon)} *du \\ &\quad + \iint_{\Omega_\epsilon} \Delta u(z) \log \frac{|z|}{R}\, dx\, dy. \end{aligned}$$

Since $\log \frac{|z|}{R}$ is integrable on $D(0,1)$, the last integral tends to

$$\iint_{D(0,R)} \Delta u(z) \log \frac{|z|}{R}\, dx\, dy$$

as $\epsilon \to 0$. It will therefore suffice to show that the second and third integrals on the right side vanish as $\epsilon \to 0$. Since u is C^2 on a neighborhood of $\overline{D(0,R)}$, the quantity $|u(z) - u(0)|$ tends to zero uniformly as $|z| \to 0$, and the magnitudes $|u_x(z)|$ and $|u_y(z)|$ of the first derivatives are uniformly bounded (say by $M > 0$) on $D(0,R)$. Hence, first of all

$$\lim_{\epsilon\to 0} \left| \int_0^{2\pi} (u(\epsilon e^{i\theta}) - u(0))\, d\theta \right| \le \int_0^{2\pi} \lim_{\epsilon \to 0} |u(\epsilon e^{i\theta}) - u(0)|\, d\theta = \int_0^{2\pi} 0\, d\theta = 0,$$

and secondly,

$$\begin{aligned} \left| \log\frac{\epsilon}{R} \int_{bD(0,\epsilon)} *du \right| &= \log\frac{R}{\epsilon} \left| \int_0^{2\pi} \frac{\partial u}{\partial n}(\epsilon e^{i\theta}) \epsilon\, d\theta \right| \\ &\le 2\pi\epsilon \left(\log \frac{R}{\epsilon} \right) \cdot \max_{|z|=\epsilon} |\nabla u(z)| \le 4\pi M \epsilon \log \frac{R}{\epsilon}, \end{aligned}$$

which tends to zero as $\epsilon \to 0$. The desired formula is now established.
□

(b) Use this formula to give a different proof that a C^2 function is subharmonic if and only if its Laplacian is non-negative everywhere.

Solution. If $\Delta u \geq 0$ everywhere, then the last integrand in the first part of this problem is non-positive everywhere. Hence

$$2\pi u(P) \leq \int_0^{2\pi} u(P + Re^{i\theta})\, d\theta,$$

whenever $\overline{D(P,R)} \subset \Omega$. That is, u has the subaveraging property and is therefore subharmonic.

If, on the other hand, $\Delta u(P) = -m < 0$ for some $P \in \Omega$, then the fact that u is C^2 implies that there exists $\delta > 0$ such that $\Delta u(z) < -m/2$ for all $P \in \overline{D(P,\delta)} \subset \Omega$. Hence the formula from the first part of this problem, and the fact that $\log \frac{|z-P|}{\delta} < 0$ on $D(P,\delta)$ gives

$$2\pi u(P) \geq \int_0^{2\pi} u(P + \delta e^{i\theta})\, d\theta - \frac{m}{2} \iint_{D(0,\delta)} \log \frac{|z-P|}{\delta}\, dx\, dy > \int_0^{2\pi} u(P + \delta e^{i\theta})\, d\theta.$$

So the subaveraging property fails at P, and u is not subharmonic on any neighborhood of P. □

Problem 2. A domain in $\Omega \subset \mathbf{C}$ is *doubly-connected* if $\mathbf{C} - \Omega$ has exactly one bounded component K (which is therefore necessarily connected and compact). Supposing that Ω is doubly-connected, find a simple domain that is biholomorphically equivalent to Ω as follows:

(a) Show that if K is a single point and/or the unbounded component of $\mathbf{C} - \Omega$ is empty, then Ω is biholomorphically equivalent to $\mathbf{C}^*$ or to $D^*(0,1)$. Then assume for the remainder of the problem that Ω is not biholomorphic to $\mathbf{C}^*$ or $D^*(0,1)$. As I explained in class, we can therefore assume that $\Omega = D(0,1) - \overline{U}$ where $U \subset D(0,1)$ is a relatively compact open set with smooth boundary bU. In particular, all points in $b\Omega$ are regular for the Dirichlet problem.

Solution. Suppose $K = \{z_0\}$ is a single point. If there are no unbounded components in $\mathbf{C} - \Omega$, then $\Omega = \mathbf{C} - \{z_0\}$ and $z \mapsto z + z_0$ sends Ω biholomorphically onto $\mathbf{C}^*$. If $\mathbf{C} - \Omega$ has an unbounded component, then $\Omega \cup \{z_0\}$ is a simply connected domain not equal to $\mathbf{C}$. Therefore the Riemann mapping theorem gives us a biholomorphism $f : \Omega \cup \{z_0\} \to D(0,1)$ that sends z_0 to 0. Restricting to Ω, we obtain a biholomorphism $f : \Omega \to D^*(0,1)$.

Finally, suppose instead that K contains more than one point, but $\mathbf{C} - \Omega$ has no unbounded components. Choosing $z_0 \in K$ and applying the transformation $z \mapsto \frac{1}{z - z_0}$ puts us in the situation we just took care of, because the point at infinity is sent to the origin which then comprises the entire bounded component of $\mathbf{C} - \Omega$. Hence Ω is biholomorphic to $D^*(0,1)$. □

(b) Let δ be a small (enough) positive number. Explain why any piecewise C^1 closed curve $\gamma \subset \Omega$ is homologous to $kbD(0, 1-\delta)$ for exactly one $k \in \mathbf{Z}$.

Solution. We know that γ is homologous to any closed curve that has the same index about every point in $\mathbf{C} - \Omega$. We proved last semester that this index is

- the same for all points in the same connected component of $\mathbf{C} - \Omega$; and (in particular)
- zero for points in unbounded components of $\mathbf{C} - \Omega$.

Thus if we fix a point $z_0 \in K$, we have that γ is homologous to any curve whose index about z_0 coincides with $k := \text{ind}_\gamma(z_0)$. The index is always an integer, and for $bD(0, 1-\delta)$, it is one (since $z_0 \in D(0, 1-\delta)$). Hence $\gamma \sim kbD(0, 1)$. □

(c) Let $h : \Omega \to \mathbf{R}$ be the harmonic extension of the function equal to 1 on $bD(0, 1)$ and to 0 on bU. Show that $\int_{bD(0,1-\delta)} *dh > 0$ and therefore that h does not have a harmonic conjugate on all of Ω. (Hint: consider the derivative of $m(r) := \int_0^{2\pi} h(re^{i\theta})\, d\theta$ with respect to r).

Solution. Note that $m(1) = 2\pi$ but $m(r) < 2\pi$ for any $r < 1$ (such that $m(r)$ is well-defined), since as a non-constant harmonic function $h(z) < 1 = \max_{w \in b\Omega} h(w)$ for all $z \in \Omega$. Now $m(r)$ is continuous for $r \leq 1$ and differentiable for $r < 1$, so by the mean value theorem, there exists $r' \in (r, 1)$ such that $(1-r)m'(r') = m(1) - m(r) > 0$. In particular, $m'(r') > 0$. On the other hand,

$$m'(r') = \int_0^{2\pi} \frac{dh}{dr}(r'e^{i\theta})\, d\theta = r' \int_{bD(0,r')} \frac{dh}{dn}(r'e^{i\theta})\, ds = r' \int_{bD(0,r')} *dh$$

by the fourth part problem 1 on the last homework. Since $bD(0, 1-\delta)$ is homologous in Ω to $bD(0, r')$, the assertion follows. □

(d) Show nevertheless that there exists (a smallest) $\alpha > 0$ and a holomorphic function $f : \Omega \to \mathbf{C}$ such that $|f| = e^{\alpha h}$.

Solution. Fix $z_0 \in \Omega$ and for any $z \in \Omega$, define

$$f(z) = \exp \alpha \left(h(z) + i \int_{z_0}^{z} *dh \right)$$

where the integral is taken over some path from z_0 to z. To the extent that the second integral in parenthesis is well-defined, it gives a harmonic conjugate for h, and f is therefore holomorphic.

Of course, we need to know that at least f (if not the harmonic conjugate of h) is well-defined. This amounts to knowing that if we choose two different paths γ_1, γ_2 from z_0 to z, then

$$\alpha \left(\int_{\gamma_1} *dh - \int_{\gamma_2} *dh \right) = 2\pi\ell$$

for some $\ell \in \mathbf{Z}$. Since $\gamma_1 - \gamma_2$ is a closed curve, we know from the first part of this problem that it is homologous to $kbD(0, 1-\delta)$. Therefore the difference above is the same as $k\alpha \int_{bD(0,1-\delta)} *dh$. This last quantity will be an integer multiple of 2π for all $k \in \mathbf{Z}$ if and only if it is when

$k = 1$. Since $I := \int_{bD(0,1-\delta)} *dh > 0$, we see that $\alpha = 2\pi/I$ is the smallest positive number that makes f well-defined. $\square$

(e) Let $\Omega_\delta = \{z \in \Omega : d(z, \mathbf{C} - \Omega) > \delta\}$. Then (you can take this for granted) $b\Omega_\delta$ is a union of two C^1 simple closed curves: $bD(0, 1-\delta)$ and another simple closed curve γ close to bU. Given $|w| > 1$, explain why the index of $f \circ \gamma$ about w is zero when δ is small enough.

Solution. By construction $\lim_{z\to bU} |e^{f(z)}| = \lim_{z\to bU} e^{h(z)} = 1$. Hence for δ small enough, we have $f \circ \gamma(z) \in D(0, |w|)$ for all $z \in \gamma$. In particular, w lies in the unbounded component of the complement of $f \circ \gamma$. It follows that $\mathrm{ind}_{f\circ\gamma}(w) = 0$. $\square$

(f) Show that when $|w| < e^\alpha$ and $\delta > 0$ is small enough, the index of $f(bD(0, 1-\delta))$ about w is equal to 1. (Hint: treat $w = 0$ first, using the argument principle and the relationship between f and h)

Solution. Suppose first that $w = 0$. Then the index of $f(bD(0, 1-\delta))$ about w is given by

$$I := \frac{1}{2\pi i} \int_{bD(0,1-\Delta)} \frac{f'(z)}{f(z)}\, dz = \int_{bD(0,1-\Delta)} g'(z)\, dz,$$

where $g = \alpha(h + ih^*)$ is only defined locally (up to an additive constant that disappears when differentiating) by choosing a harmonic conjugate h^* for h. Since g is holomorphic, we have that.

$$dg = \frac{\partial g}{\partial z}\, dz + \frac{\partial g}{\partial \bar z}\, d\bar z = g'(z)\, dz.$$

Hence $g'(z)\, dz = \alpha(dh + id * h) = \alpha(dh + i * dh)$, and we can continue to compute

$$2\pi i I = \alpha \int_{bD(0,1-\delta)} dh + i * dh = i\alpha \int_{bD(0,1-\delta)} *dh = 2\pi i.$$

The second equality follows because the integral of an exact 1-form about a closed curve is zero. The third equality is a consequence of my choice of α. Thus the index $I = 1$ as asserted.

Now assume only that $|w| < e^\alpha$. Since

$$\lim_{z\to bD(0,1)} |e^{f(z)}| = \lim_{z\to bD(0,1)} e^{\alpha h(z)} = e^\alpha,$$

we can choose $\delta > 0$ small enough so that $|f(z)| > |w|$ for all $z \in bD(0, 1-\delta)$. In particular, w and 0 lie in the same component of the complement of $f(bD(0, 1-\delta))$, and the index of $f(bD(0, 1-\delta))$ about w is the same as the index about 0, which we just computed to be 1. $\square$

(g) Conclude that f maps Ω biholomorphically onto the annulus $A = \{1 < |w| < e^\alpha\}$.

Solution. The maximum principle (applied to h and $-h$) tells us that $0 < h(z) < 1$ for all $z \in \Omega$. Hence $1 \leq e^\alpha h(z) = |f(z)| \leq e^\alpha$ for all $z \in \Omega$. That is, $f(\Omega) \subset A$. On the other hand, given $w \in A$, the number (counting multiplicity, as always) of f-preimages of w in Ω_δ is equal to the index of $f(b\Omega_\delta)$ about w. This is the difference between the indices about w of $f(bD(0, 1-\delta)$ and $f(\gamma)$, which for small δ we showed to be

$1 - 0 = 1$. Letting $\delta \to 0$, we conclude that each $w \in A$ has exactly one f-preimage in Ω. So f is a holomorphic bijection from Ω onto A and therefore a biholomorphism. $\square$