Assignment 4
Math 60380, Winter '10
Due Friday, April 2

**Problem 1.** (Linearizing coordinates near at attracting fixed point) Let $P : \mathbf{C} \to \mathbf{C}$ be a polynomial function (this would work for rational functions, too, but then the point at infinity might become involved, holomorphic things would become meromorphic, etc) and $z_0 = P(z_0)$ be an attracting fixed point. That is, $P'(z_0) = \lambda$ with $|\lambda| < 1$. Let $A$ be the immediate basin of $z_0$. Then there is a surjective holomorphic function $\psi : A \to \mathbf{C}$ such that $\psi(z_0) = 0$, $\psi'(z_0) = 1$ and $\psi \circ P = \lambda\psi$. Hence $w = \psi(z)$ conjugates the map $z \mapsto P(z)$ on a small neighborhood of $z_0$ to the linear transformation $w \to \lambda w$ near 0. Prove the existence of $\psi$ as follows:

- Without loss of generality (i.e. by conjugating with an affine transformation) $z_0 = 0$.

  *Solution.* Let $\varphi(z) = z + z_0$ and $\tilde{P} = \varphi^{-1} \circ P \circ \varphi$. Then since $\varphi(0) = z_0$, we see that $\tilde{P}$ has a fixed point at 0, and since multipliers of fixed points are unchanged by holomorphic conjugacy, 0 is attracting for $\tilde{P}$ with multiplier $\lambda$. □

- Given a positive number $r' \in (|\lambda|, 1)$, there exists $\epsilon > 0$ such that $z \in D(0, \epsilon)$ implies $|P^n(z)| \leq (r')^n|z|$.

  *Solution.* Assuming (as I will from now on) that $z_0 = 0$, I have $P(z) = \lambda z + z^2 Q(z)$ for some other polynomial $Q$. Let $M = \max_{z \in D(0,1)} |Q(z)|$. Given any number $r' \in (|\lambda|, 1)$, I choose $\epsilon = \min\{1, \frac{r'-|\lambda|}{M}\}$. Then if $z \in D(0, \epsilon)$, I have

$$|P(z)| \leq |\lambda||z| + |z|^2|Q(z)| \leq (|\lambda| + M|z|)|z| < \left(|\lambda| + M\frac{r' - |\lambda|}{M}\right)|z| = r'|z|.$$

  In particular, since $r' < 1$, I see that $z \in D(0, \epsilon)$ implies that $P(z) \in D(0, \epsilon)$ and hence inductively $P^n(z) \in D(0, \epsilon)$ for every $n \in \mathbf{N}$. In particular, I can iterate my estimate on $|P(z)|$ and obtain $|P^n(z)| < (r')^n|z|$ for every $n \in \mathbf{N}$. □

- Let $\psi_n(z) = \lambda^{-n}P^n(z)$. Show that there exists $\epsilon > 0$ such that $\psi_n$ converges uniformly on $D(0, \epsilon)$ by estimating $|\psi_j(z) - \psi_{j-1}(z)|$ and using $\psi_n = \psi_0 + \sum_{j=1}^{n}(\psi_j - \psi_{j-1})$.

  *Solution.* Observe that

$$|\psi_j(z) - \psi_{j-1}(z)| = |\lambda^{-j}P^j(z) - \lambda^{-j+1}P^{j-1}(z)| = \frac{|P(w) - \lambda w|}{|\lambda^j|} = \frac{|w^2 Q(w)|}{|\lambda|^j},$$

  where $w = P^{j-1}(z)$ and $Q$ is the polynomial introduced in my solution to the previous part of this problem. So if I choose $r' \in (|\lambda|, 1)$ so that $(r')^2 < |\lambda|$, then the previous part of the problem gives me an $\epsilon > 0$ such that $z \in D(0, \epsilon)$ implies $|P^{j-1}(z)| < (r')^{j-1}|z| < \epsilon(r')^{j-1}$ for all $j \geq 1$. Plugging into the previous estimate gives

$$|\psi_j(z) - \psi_{j-1}(z)| \leq M\frac{|P^{j-1}(z)|^2}{|\lambda|^j} \leq \frac{M\epsilon^2}{r'} \cdot \left(\frac{(r')^2}{|\lambda|}\right)^j = Ct^j$$

for all $z \in D(0, \epsilon)$ and all $j \geq 1$, where $C > 0$ and $t = (r')^2/|\lambda| < 1$. Hence the series

$$\sum_{j=1}^{\infty}(\psi_j(z) - \psi_{j-1}(z))$$

is dominated by the convergent geometric series $\sum Ct^j$ and therefore converges absolutely and uniformly on $D(0, \epsilon)$. It follows that the functions $\psi_n = \psi_0 + \sum_{j=1}^{n}(\psi_j - \psi_{j-1})$ converge uniformly on $D(0, \epsilon)$. □

- Let $\psi = \lim \psi_n$. Show that $\psi$ and its first derivative take the correct values at 0 and that $\psi \circ P = \lambda\psi$ on $D(0, \epsilon)$.

*Solution.* Clearly $\psi(0) = \lim \lambda^{-n} P^n(0) = \lim \lambda^{-n} \cdot 0 = 0$. Also, since the $\psi_n$ are holomorphic functions converging uniformly on $D(0, \epsilon)$, the sequence $\psi_n'$ of derivatives converges normally, too. Hence

$$\psi'(0) = \lim \psi_n'(0) = \lim \lambda^{-n}(P^n)'(0) = \lim \lambda^{-n} \cdot \lambda^n = \lim 1 = 1.$$

Finally, I have

$$\psi \circ P = \lim \psi_n \circ P = \lim \lambda^{-n} P^{n+1} = \lambda \lim \psi_{n+1} = \lambda\psi,$$

as desired. □

- Use the relationship between $\psi$ and $P$ to extend the definition of $\psi$ to all of $A$. Show that $\psi(A) = \mathbf{C}$.

*Solution.* The iterates of $P$ converge uniformly to the constant function $z \mapsto 0$ on any compact subset $K \subset A$. So given such a $K$, there exists $N \in \mathbf{N}$ so that $n \geq N$ implies $|P^n(z)| < \epsilon$ for all $z \in K$. I therefore define $\psi(z) = \lambda^{-n}\psi \circ P^n(z)$ for all $z \in K$ and some $n \geq N$. To see that this definition is consistent, I need to know that it is independent of my choice of $n$. So consider another index $m \geq N$; without loss of generality, say $m > n$. Then since $P^n(z) \in D(0, \epsilon)$, I have from the previous part of this problem that

$$\lambda^{-m}\psi \circ P^m(z) = \lambda^{-n}\lambda^{n-m}\psi \circ P^{m-n}(P^n(z)) = \lambda^{-n}\psi(P^n(z)),$$

which shows that my definition of $\psi$ on $K$ does not depend on my choice of index. Exhausting $A$ with compact sets $K$, I can therefore extend the definition of $\psi$ to the entire immediate basin $A$ of 0. Since $\psi \circ P^n$ varies holomorphically with $z$, the extended function $\psi$ is holomorphic in $z$.

In particular, since $\psi \circ P - \lambda\psi \equiv 0$ on $D(0, \epsilon)$, I have $\psi \circ P = \lambda\psi$ on all of $A$. To see that $\psi(A) = \mathbf{C}$, note that $D(0, \epsilon) \subset P^{-n}(D(0, \epsilon))$ for all $n \in \mathbf{N}$. So if $A_n \subset \mathbf{C}$ is the connected component of $P^{-n}(D(0, \epsilon))$ containing $D(0, \epsilon)$ it follows from backward invariance of the Fatou set and the definition of immediate basin that $A_n \subset A$ and $P^n(A) = D(0, \epsilon)$. Therefore $\psi(A_n) = \lambda^n\psi(P^n(A_n)) = \lambda^n\psi(D(0, \epsilon))$ for every $n \in \mathbf{N}$. Since $\psi$ is non-constant, we have that $\psi(D(0, \epsilon)$ contains a disk $D(0, \delta)$ about $0 = \psi(0)$. Therefore the sets $\psi(A_n) \supset D(0, \lambda^n\delta)$ exhaust $\mathbf{C}$. □

- Recall that the Julia set of $P$ is infinite when $\deg P \geq 2$. Show in this case that $\psi$ cannot be injective. Show further that $\psi$ has a critical point at $z \in A$ if and only if $P^n(z)$ is critical for $P$ for some $n \geq 0$.

*Solution.* If $\psi : A \to \mathbf{C}$ were injective, it would be biholomorphic, and since $\psi(A) = \mathbf{C}$, the inverse function would be a holomorphic function $\psi^{-1} : \mathbf{C} \to A$. But $A$ is disjoint from the Julia set of $P$, and since the Julia set is infinite, $A$ omits at least two points in $\mathbf{C}$. So by Picard's little theorem $\psi$ is constant, contradicting the fact that $\psi'(0) = 1$. It follows that $\psi$ is not injective.

Since $\psi'(0) \neq 0$, we may assume (shrinking $\epsilon$ if necessary) that $\psi'(z) \neq 0$ for all $z \in D(0, \epsilon)$. Now if $z \in A$ is any point in the immediate basin of 0, and $n \in \mathbf{N}$ is large enough that $w := P^n(z) \in D(0, \epsilon)$, then we have $\psi'(z) = \lambda^{-n}(\psi \circ P^n)'(z) = \lambda^{-n}\psi'(w)(P^n)'(z) = 0$ if and only if $(P^n)'(z) = 0$. But $0 = (P^n)'(z) = P'(P^{n-1}(z)) \cdot \dots \cdot P'(z)$ if and only if $P'(P^j(z)) = 0$ for some $j \in \{0, \dots, n-1\}$. □

**Problem 2.** Recall the construction of the complex torus from class: beginning with $\omega_1, \omega_2 \in \mathbf{C}$ not on the same line, one defines $\Gamma = \{n_1\omega_1 + n_2\omega_2 : n_j \in \mathbf{Z}\}$ and $R = \mathbf{C}/\Gamma$ with the quotient topology. A key fact is that $\Gamma \cap D(0, \epsilon) = \{0\}$ for $\epsilon > 0$ small enough. Show the following additional facts:

- $\Gamma$ is discrete and closed.

*Solution.* Suppose, to get a contradiction, that $z \in \mathbf{C}$ is a limit point of $\Gamma$. Given $\epsilon > 0$ as above, the disk $D(z, \epsilon/2)$ must contain infinitely many distinct points in $\Gamma$. If $\omega, \omega' \in \Gamma$ are two of these, then $|\omega - \omega'| \leq |\omega - z| + |\omega' - z| < \epsilon/2 + \epsilon/2 = \epsilon$. Writing $\omega = n_1\omega_1 + n_2\omega_2$, $\omega' = n_1'\omega_1 + n_2'\omega_2$, we see that $\omega - \omega' = (n_1 - n_1')\omega_1 + (n_2 - n_2')\omega_2$ is a non-zero point in $\Gamma \cap D(0, \epsilon)$, contradicting our choice of $\epsilon$. This proves that $\Gamma$ has no limit points in $\mathbf{C}$—i.e. $\Gamma$ is closed and discrete. □

- The quotient map $\pi : \mathbf{C} \to R$ is a covering.

*Solution.* Let $p = \pi(z) \in R$ be any point and $U = \pi(D(z, \epsilon/2))$ (again with $\epsilon > 0$ as above). Then $\pi^{-1}(U) = \bigcup_{\omega \in \Gamma} D(z + \omega, \epsilon/2)$. Since $\pi$ is continuous and open by definition, it suffices to show that the disks $D(z + \omega, \epsilon/2)$ in this union are mutually disjoint and that the restriction of $\pi$ to any one of them is injective. Note that if $w \in D(z + \omega, \epsilon/2) \cap D(z + \omega', \epsilon/2)$ lies in two different disks, then

$$|\omega - \omega'| = |(\omega + z) - (\omega' + z)| \leq |(\omega + z) - w| + |(\omega' + z) - w| < \epsilon/2 + \epsilon/2 = \epsilon,$$

again contradicting our choice of $\epsilon$. Hence the disks $D(z + \omega, \epsilon/2)$ are mutually disjoint. Similarly, if $\pi(w) = \pi(w')$ for two points $w, w' \in D(z + \omega, \epsilon/2)$, then $w - w' \in \Gamma \cap D(0, \epsilon$ which implies that $w = w'$. Hence $\pi|_{D(z+\omega,\epsilon/2)}$ is injective. □

- $R$ is Hausdorff.

*Solution.* Let $p = \pi(z), p' = \pi(z') \in R$ be two distinct points. Then $\pi^{-1}(p') = \{z' + \omega : \omega \in \Gamma\}$, which is closed in $\mathbf{C}$. Since $\{z\}$ is compact and not contained in $\pi^{-1}(p')$, the distance $\delta$ from $z$ to $\pi^{-1}(p')$ is positive. I claim that the neighborhoods $U = \pi(D(z, \delta/2)$ and $U' = \pi(D(z', \delta/2))$ of $p$ and $p'$, respectively, are disjoint. If $q \in U \cap U'$ lay in both sets, then $q = \pi(\zeta) = \pi(\zeta')$ for some $\zeta \in D(z, \delta/2), \zeta' \in D(z', \delta/2)$. Thus $\zeta - \zeta' \in \Gamma$ and $|z - (z' + \omega)| \leq |z - \zeta| + |z - \zeta'| < \delta$ contradicting the fact that $z$ is at distance $\delta$ from $\pi^{-1}(p')$. $\square$

- $R$ is compact.

*Solution.* Since $\omega_1, \omega_2$ are linearly independent over $\mathbf{R}$, we can write any point $z \in \mathbf{C}$ as $z = x_1\omega_1 + x_2\omega_2 = (n_1 + r_1)\omega_1 + (n_2 + r_2)\omega_2$ where $x_j \in \mathbf{R}$, $n_j \in \mathbf{Z}$, $r_j \in [0, 1)$. Hence $\pi(z) = \pi(r_1\omega_1 + r_2\omega)2) \in R$. It follows that $R = \pi(\mathbf{C})$ is the continuous image of the compact set $\{r_1\omega_1 + r_2\omega_2 : r_j \in [0, 1]\}$. Thus $R$ is compact. $\square$

**Problem 3.** Let $R$ and $S$ be Riemann surfaces. A continuous map $f : R \to S$ is *proper* if the preimage $f^{-1}(K)$ of every compact set $K \subset S$ is compact in $R$ (Note that if $R$ itself is compact, this condition is trivially satisfied). Suppose now that $f$ is holomorphic, proper, and non-constant. For any $p \in S$, let $\deg_f(p)$ be the number of preimages of $p$ counted with multiplicity.

- Give equivalent definitions of the *multiplicity* of a preimage of $p$ from two points of view: the order of vanishing of a holomorphic function and something more topological. You don't have to prove equivalence.

*Solution.* Suppose $f(q) = p$. Let $z : U \subset R \to \mathbf{C}$ and $w : V \subset R \to \mathbf{C}$ be charts about $q$ and $p$. Adding constants to $z$ and $w$, I may assume $z(q) = 0 = w(p)$. The function $h = w \circ f \circ z^{-1}$ is therefore defined on a neighborhood of 0 and satisfies $h(0) = 0$. I define the multiplicity of $f$ at $p$ to be the order of vanishing of $h$ at 0. Alternatively, $m$ is multiplicity of $q$ as a preimage of $p$ if for any neighborhood $U \ni q$ such that $\overline{U} \cap f^{-1}(p) = \{q\}$, there is a neighborhood $W \ni p$ such that $m = \#f^{-1}(p') \cap U$ for all $p' \in W \setminus \{p\}$.

- Show that $\deg_f(p)$ is finite for all $p \in S$.

*Solution.* Since $\{p\}$ is compact and $f$ is proper, so is $f^{-1}(p)$. So if the set $f^{-1}(p)$ is infinite, it must have an accumulation point, which by the identity theorem implies that $f$ is constant. Hence $f^{-1}(p)$ is finite. Now $\deg_f(p) = \sum_{q \in f^{-1}(p)} m(q)$, where $m(q)$ is the multiplicity of $q$ as a preimage of $p$. The order of vanishing definition for multiplicity makes clear that $m(q)$ is finite when $f$ is non-constant. Combined with finiteness of $f^{-1}(q$, we infer that $\deg_f(p)$ is finite. $\square$

- Let $S_n = \{p \in S : \deg_f(p) \geq n\}$. Show that $S_n$ is open.

*Solution.* The second definition of multiplicity for preimages makes clear that if $f(q) = p$ with multiplicity $m$, then there is a neighborhoods $W \ni p$ such that $f^{-1}(p')$ contains $m$ points near $q$ for all $p' \in W$. Since there are only finitely many preimages of $p$, we can choose a single neighborhood $W$ that works for all $q \in f^{-1}(p)$ simultaneously. Hence $p' \in W$ implies that $\#f^{-1}(p') \geq \sum_{q \in f^{-1}(p)} m(q) = \deg_f(p)$. So if $p \in S_n$, so is $W$, which proves that $S_n$ is open. $\square$

- Let $(p_j) \subset S_n$ be a sequence of points converging to $p \in R$. Suppose that no $p_j$ is a critical value of $f$ (i.e. $p_j = f(q)$ with multiplicity larger than one.) Show that after refining the sequence, if necessary, $\deg_f(p_j)$ converges and $\lim \deg_f(p_j) \le \deg_f(p)$. (Hint: $K = \{p_j : j \in \mathbf{N}\} \cup \{p\}$ is compact).

  *Solution.* Since $p_j \in S_n$ and $p_j$ is not a critical value, we can for each $j$ choose $n$ distinct preimages $q_{1j}, \dots, q_{nj} \subset f^{-1}(p_j)$. As noted, $K$ is compact. So by proper therefore $f^{-1}(K)$ is, too. This means that after refining the given sequence, we may assume for each $1 \le i \le n$ that $q_{ij} \to q_i \in R$. By continuity $f(q_i) = p$. So if all the $q_i$ are distinct, it follows that $\deg_f(p) \ge n = \#\{q_i\}$. On the other hand, if, say $k$ of the $q_i$ are the same point $q$, then any neighborhood $U \ni q$ will contain at least $k$ preimages of $p_j$ for $j$ large enough. By the second definition of multiplicity, we infer that $q$ has multiplicity $\ge k$ as a preimage of $p$. So even if the $q_i$ are not distinct, it follows that $\deg_f(p) \ge n$.

- Conclude that $S = S_n$ for $n$ small enough and that in fact, $\deg_f(p)$ is independent of $p$. The number $\deg_f(p)$ is therefore simply called the *degree of* $f$. This is a generalization of the notion of the degree of a rational function $R : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$

  *Solution.* I claim that the previous part of this problem implies that $S_n$ is closed. Indeed this is immediate once I show that no point in $S$ is a limit of critical values of $f$. Then in particular any limit point $p$ of $S_n$ is a limit of a sequence of non-critical values, and the previous item shows that $p \in S_n$.

  So assume to get a contradiction that $p$ is a limit of critical values $(p_j) \subset S$. That is $p_j = f(q_j)$ where $q_j$ is a critical point. As in the previous item, I can assume that $q_j \to q$ where $f(q) = p$. Now I choose coordinate charts $z : U \ni q \to \mathbf{C}$ and $w : V \ni p \to \mathbf{C}$ such that $z(q) = 0 = w(p)$ and consider the holomorphic function $h = w \circ f \circ z^{-1}$ defined near 0. Without loss of generality, $q_j \in U$, $p_j \in V$, and $h$ is defined at $z_j := z(p_j)$. Then $z_j \to 0$ and $h'(z_j) = 0$ for all $j$. It follows that $h' \equiv 0$ near 0, so that $h$ is constant near 0. Equivalently $f$ is constant near $q$ which implies by the identity theorem that $f$ is constant, contrary to assumption.

  Finally, as an open and closed subset of $S$, the set $S_n$ is either empty or equal to $S$. It is certainly non-empty for $n = 1$ small enough by the first part of this problem, so $S_n = S$ for $n$ small enough. Now if $n$ is the largest $n$ for which $S_n = S$, it follows that all points in $S$ have at least $n$ preimages. On the other hand $S_{n+1}$ must be empty, so no point has more than $n$ preimages. I conclude that $\deg_f(p) = n$ for all $p \in S$. $\square$

**Problem 4.** (Added 4/13/10) Many facts about harmonic functions can be established by recourse to similar facts about holomorphic functions. For instance, if $\Omega \subset \mathbf{C}$ is a domain in the complex plane (or more generally, a Riemann surface) and $h : \Omega \to \mathbf{R}$ is harmonic, then it is often quite useful to find a holomorphic function $f : R \to \mathbf{C}$ such that $|f| = e^h$. Use this idea to prove each of the following statements:

(a) Let $h_n : \Omega \to [0, \infty)$ be a sequence of non-negative harmonic functions. Show that by passing to a subsequence, one can arrange that $h_n \to h$ uniformly on compacts subsets, where $h$ is either identically $\infty$ or harmonic.

*Solution.* By problem 1 on homework 4 from last semester, we may suppose that $\Omega = D(P, R)$ is a disk. Then there are holomorphic functions $g_n : \Omega \to \mathbf{C}$ such that $\operatorname{Re} g = h$. Setting $f_n = e^{g_n}$, I obtain that $|f| = e^{h_n} \geq 1$ for all $n$. Since the range of $f_n$ always omits $D(0, 1)$, it follows from Montel's Theorem that on passing to a subsequence, we have that $f_n \to f$ uniformly on compacts where $f$ is either holomorphic or $f \equiv \infty$. So if $h = \log |f|$, then $h$ is harmonic in the first case and $h \equiv \infty$ in the second. Moreover, $|f_n| = e^{h_n}$ converges uniformly on compacts to $|f| = e^h$. Since $|x - y| \leq |e^x - e^y|$ for all $x, y \geq 1$, we have $|h_n - h| \leq |e^{h_n} - e^h|$, which implies that $h_n \to h$ uniformly on compacts.

Now when $\Omega$ is not a disk, I can at least choose countably many disks $D_j = D(P_j, R_j) \subset \Omega$ that cover $\Omega$. By refining the sequence $(h_n)$ to get a normally convergence subsequence $(h_{1n})$ on $D_1$ and then refining $(h_{1n})$ to get a normally convergent subsequence $(h_{2n})$ on $D_2$, etc, I arrive at a diagonal subsequence $(h_{nn})$ that converges normally on every disk $D_n$ in my cover.

(b) Let $h : D^*(0, 1) \to \mathbf{R}$ be a harmonic function with an isolated singularity at 0. Show that if $|h(z)| \leq M$ for all $z \in D^*(0, 1)$, then $h$ extends harmonically past 0.

*Solution.* Since $D^*(0, 1)$ is doubly connected, problem 2d from Homework 2 gives us a number $\alpha > 0$ and a holomorphic function $f : D^*(0, 1) \to \mathbf{C}$ such that $|f| = e^{\alpha h}$. In particular, $|f| \leq e^{\alpha M}$ on $D^*(0, 1)$. So by Riemann's removable singularity theorem, $f$ extends to a holomorphic function $f : D(0, 1) \to \mathbf{C}$. The bound on $h$ also implies that $e^f \geq e^{-\alpha M} > 0$ on $D^*(0, 1)$, so it follows that $f(0) \neq 0$. Hence $h$ extends harmonically past 0 by setting $h(0) = \alpha^{-1} \log |f(0)|$.

Note that since the domains in question are not (necessarily) simply connected in either statement, one cannot automatically assume the existence of a harmonic conjugate for the given function $h$. In proving the first assertion, you can skirt this issue by taking advantage of problem 1 on homework 4 from last semester (it's stated for holomorphic functions, but that's actually irrelevant to the argument used to prove it). In the second assertion, you can imitate the idea from problem 2d on Homework 2, taking advantage of the fact that $D^*(0, 1)$ is doubly-connected.