EQUIDISTRIBUTION OF FEKETE POINTS ON COMPLEX MANIFOLDS

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Abstract. We prove the several variable version of a classical equidistribution theorem for Fekete points of a compact subset of the complex plane, which settles a well-known conjecture in pluri-potential theory. The result is obtained as a special case of a general equidistribution theorem for Fekete points in the setting of a given holomorphic line bundle over a compact complex manifold. The proof builds on our recent work "Capacities and weighted volumes for line bundles".

1. Introduction

A classical potential theoretic invariant of a compact set E in the complex plane $\mathbf{C}$ is given by its *transfinite diameter:*

$$d_\infty(E) := \lim_{k\to\infty}\left(\sup_{z_0,...,z_k\in E}\prod_{0\leq i<j\leq k}|z_i - z_j|\right)^{2/k(k-1)}. \qquad (1.1)$$

i.e. the asymptotic geometric mean distance of points in E. A configuration $z^{(k)} \in E^{N_k}$ of points achieving the supremum for a fixed k is called a k-*Fekete configuration* for E (in classical terminology the corresponding points $(z_i^{(k)})$, for k fixed, are called *Fekete points*). A basic classical theorem (see [9] for a modern reference and [6] for the relation to Hermitian random matrices) asserts that Fekete configurations equidistribute on the *equilibrium measure* μ_E of E, i.e.

$$\frac{1}{k+1}\sum_{i=1}^{N_k}\delta_{z_i^{(k)}} \to \mu_E$$

as $k \to \infty$. In the logarithmic pluripotential theory in $\mathbf{C}^n$ a higher dimensional version of the transfinite diameter was introduced by Leja in 1959

$$d_\infty(E) := \limsup_{k\to\infty}\left(\sup_{z_1,...,z_{N_k}\in E}|\Delta(z_1, ..., z_{N_k})|\right)^{(n+1)!/nk^{n+1}}, \qquad (1.2)$$

where $\Delta(z_1, ..., z_{N_k})$ is the following higher-dimensional Vandermonde determinant:

$$\Delta(z_1, ..., z_{N_k}) := \det(e^\alpha(z_j))_{|\alpha|\leq k, 1\leq j\leq N_k},$$

where $e^\alpha(x) = x_1^{\alpha_1}...x_n^{\alpha_n}$, $\alpha \in \mathbf{N}^n$ denote the monomials and N_k is the number of monomials of degree at most k (so that $N_k = k^n/n! + O(k^{n-1})$). It was

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shown by Zaharjuta [11] that the limsup in formula (1.2) is actually a true limit, thereby answering a conjecture by Leja. However, the corresponding convergence of higher-dimensional "Fekete configurations" (towards the pluripotential theoretic equilibrium measure of E) has remained an open problem (see the survey [8] by Levenberg on approximation theory in $\mathbf{C}^n$, where it is pointed out (p. 120) that "to this date, *nothing* is known if $n > 1$").

There is also a weighted version of the previous setting where the set E is replaced by the *weighted set* (E, ϕ), with ϕ is a continuous function on the compact set E (see the appendix by Bloom in [9]). The weighted transfinite diameter $d_\infty(E, \phi)$ is then obtained by replacing $|\Delta(z_1, ..., z_{N_k})|$ in (1.2) by its weighted counterpart

$$|\Delta(z_1, ..., z_{N_k})| \, e^{-k(\phi(z_1)+...+\phi(z_N))}.$$

Even more generally, there is a variant of this weighted setting where E may be assumed to be an unbounded set in $\mathbf{C}^n$ provided that ϕ growths sufficiently fast at infinity (cf. the appendix by Bloom in [9]).

The aim of this note is to prove a generalized version of the conjecture referred to above in the more general setting of a big line bundle L over a complex manifold X (recall that from an analytic perspective L is big iff it admits a singular Hermitian metric with strictly positive curvature current). In this setting the role of $e^{-\phi}$ is played by a Hermitian metric on L (we will call the additive object ϕ a *weight* - see [3] for further notation). A weighted subset (E, ϕ) will thus consist of a subset E of X and a continuous weight on L. The "classical setting" above corresponds to the hyperplane line bundle $\mathcal{O}(1)$ over the complex projective space $\mathbf{P}^n$ with E a compact set in the affine piece $\mathbf{C}^n$ and the space $H^0(kL)$ of global holomorphic sections with values in kL (the k-th tensor power of L, written in additive notation) may in this case be identified with the space of all polynomials on $\mathbf{C}^n$ of total degree at most k.

In order to state the theorem we first recall some further notation, mostly taken from [3]. Fix a compact subset E of X which is not locally pluripolar and a continuous weight ϕ on the line bundle $L \to X$. The *equilibrium weight* of (E, ϕ) is defined by

$$\phi_E = \sup \{\varphi, \, \varphi \, \text{psh weight on} \, L, \, \varphi \leq \phi \text{ on} \, E\}, \tag{1.3}$$

where a weight φ on L is called *psh* if its curvature $dd^c\varphi$ is a positive current. In the $\mathbf{C}^n$-case above, ϕ_E is usually referred to as the *weighted Siciak extremal function* of (E, ϕ) (cf. e.g. the appendix by Bloom in [9]). The *equilibrium measure* of the weighted set (E, ϕ) is then defined as the Monge-Ampère measure

$$\mu_{(E,\phi)} := \mathrm{MA}\,(\phi_E^*), \tag{1.4}$$

of the psh weight with minimal singularities ϕ_E^* (the usc envelope of ϕ_E), that is the trivial extension to X of the Bedford-Taylor wedge-product $(dd^c\phi_E^*)^n$ computed on a Zariski open subset where ϕ_E^* is locally bounded (see [3]).

Next let $(s_1, ..., s_N)$ be a basis of $H^0(L)$ and consider the following Vandermonde-type determinant

$$\det(s)(x_1, \ldots, x_N) := \det(s_i(x_j))_{i,j}$$

which is a holomorphic section of the pulled-back line bundle $L^{\boxtimes N}$ over the N-fold product X^N.

A configuration of points $P = (x_1, ..., x_N) \in E^N$ is called a *Fekete configuration* for the weighted subset (E, ϕ) if it realizes the supremum on E^N of the pointwise length $|\det(s)|_\phi$ of $\det(s)$ with respect to the metric induced by ϕ. It is a basic observation that the definition of a Fekete configuration is actually independent of the choice of the basis (s_i). Indeed, since the top exterior power of the vector space $H^0(L)$ is one-dimensional, replacing s_i by s_i' gives $\det(s') = c\det(s')$, where c is a non-zero complex number.

As a last piece of notation, given a configuration $P = (x_1, ..., x_N) \in X^N$ for some N, we will denote by

$$\delta_P := \frac{1}{N}\sum_{i=1}^{N} \delta x_i$$

the associated probability measure.

Theorem 1.1. *Let $L \to X$ be a big line bundle over a compact complex manifold. Assume that $P_k \in X^{N_k}$ is a Fekete configuration for $(E, k\phi)$ for each k large enough. Then this sequence of configurations equidistributes towards the equlibrium measure of (E, ϕ), i.e.*

$$\lim_{k\to\infty} \delta_{P_k} = M^{-1}\mu_{(E,\phi)}$$

weakly as measures on X, where M is the total mass of $\mu_{(E,\phi)}$.

Here $N_k := h^0(kL) \simeq \mathrm{vol}(L)k^n/n!$, where $\mathrm{vol}(L) > 0$ is the *volume* of the big line bundle L, which is also equal to the total mass M of $\mu_{(E,\phi)}$ (see [3]).

In the one dimensional case, i.e. when X is a complex curve and if (E, ϕ) is taken as $(E, 0)$ where E is a compact set contained in an affine piece of X where L has been trivialized, the theorem was obtained, using completely different methods, by Bloom and Levenberg [5] (see also [7] for the case when X has genus zero and for a discussion of related interpolation problems in $\mathbf{R}^n$.)

The proof of Theorem 1.1 builds on our previous work [3], where it was shown that (minus the logarithm of) the transfinite diameter of (E, ϕ), considered as a functional on the affine space of all continuous weights on L (for E fixed) is Fréchet differentiable: its differential at the weight ϕ may be represented by the corresponding equilibrium measure $\mu_{(E,\phi)}$. A similar argument was used very recently by the first author and Witt Nystrm in [4] to obtain very general convergence results for "Bergman measures" (corresponding to Christoffel-Darboux functions in classical terminology). In fact, a unified treatment of these two convergence results can be given, which also includes the equidistribution of generic points of "small height" on an arithmetic variety obtained in [3] (which in turn generalizes Yuan's arithmetic equidistribution theorem [10]). This will be further investigated elsewhere.

Remark 1.2. In the case that L is ample the differentiability property referred to above can also be deduced from the Bergman kernel asymptotics for smooth weights in [2] as explained in section 1.4 in [3] (see also remark 11.2 in [3]). For an essentially elementary proof of these asymptotics in the weighted case in $\mathbf{C}^n$ see [1].

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1.1. **Proof of Theorem 1.1.** Given a basis $s = (s_1, ..., s_N)$ of $H^0(L)$, set

$$D_\phi := \log |\det(s)|_\phi,$$

so that D_ϕ achieves its maximum on E^N exactly at Fekete configurations of (E, ϕ) by definition. As noticed above, D_ϕ only depends on the choice of the basis s up to an additive constant. If $P = (x_1, ..., x_N) \in X^N$ is a given configuration, the more explicit formula

$$D_\phi(P) = \log |\det(s_i(x_j)| - (\phi(x_1) + ... + \phi(x_N))$$

clearly shows that $\phi \mapsto D_\phi(P)$ is an affine function, with linear part given by integration against $-N\delta_P$.

In order to normalize D_ϕ, we fix an auxiliary (smooth positive) volume form μ on X and smooth metric $e^{-\psi}$ on L and take $(s_1, ..., s_N)$ to be an orthonormal basis of $H^0(L)$ with respect to the corresponding L^2 scalar product. It is easily seen (cf. [3]) that D_ϕ becomes independent of the choice of such an orthonormal basis. The main result of [3] implies that

$$\lim_{k\to\infty} \frac{(n+1)!}{k^{n+1}} \sup_{E^{N_k}} D_{k\phi} = \mathcal{E}(\psi_X, \phi_E^*) \tag{1.5}$$

where ψ_X is the equilibrium weight of (X, ψ) and $\mathcal{E}$ denotes the Aubin-Yau energy, whose precise formula doesn't matter here. The main point for what follows is that

$$\phi \mapsto \mathcal{E}(\psi_X, \phi_E^*)$$

is Fréchet differentiable on the space of all continuous weights on L, with derivative at ϕ in the tangent direction $v \in C^0(X)$ given by

$$(n+1) \int_X v \, \mu_{(E,\phi)}.$$

This is indeed the content of Theorem 5.7 of [3]. Now let $P_k \in E^{N_k}$ be a sequence of configurations, and set

$$F_k(\phi) := -\frac{1}{kN_k} D_{k\phi}(P_k)$$

and

$$G(\phi) := \frac{1}{(n+1)M} \mathcal{E}(\phi_E^*, \psi_X).$$

We thus see that

$$\liminf_{k\to\infty} F_k(\phi) \geq G(\phi),$$

and furthermore

$$\lim_{k\to\infty} F_k(\phi) = G(\phi)$$

if $P_k \in E^{N_k}$ is a Fekete configuration for $(E, k\phi)$, as follows from (1.5) and the fact that $N_k \simeq Mk^n/n!$.

As noticed above, the functional F_k is affine, with linear part given by integration against δ_{P_k}. On the other hand the differentiability property of the energy writes

$$\frac{d}{dt}_{t=0} G(\phi + tv) = M^{-1} \int_X v\, \mu_{(E,\phi)}.$$

The proof of Theorem 1.1 is thus concluded by the following elementary result applied to $f_k(t) := \mathcal{F}_k(\phi + tv)$ and $g(t) := G(\phi + tv)$, taking $P_k \in E^{N_k}$ to be a Fekete configuration for $(E, k\phi)$ as in the Theorem.

Lemma 1.3. *Let f_k by a sequence of concave functions on $\mathbf{R}$ and let g be a function on $\mathbf{R}$ such that*

- $\liminf_{k\to\infty} f_k \geq g$.
- $\lim_{k\to\infty} f_k(0) = g(0)$.

If the f_k and g are differentiable at 0, then

$$\lim_{k\to\infty} f_k'(0) = g'(0).$$

Proof. Since f_k is concave, we have

$$f_k(0) + f_k'(0)t \geq f_k(t)$$

and it follows that

$$\liminf_{k\to\infty} t f_k'(0) \geq g(t) - g(0).$$

The result now follows by first letting $t > 0$ and then $t < 0$ tend to 0. □

The same lemma underlies the proof of Yuan's equidistribution theorem given in [3]. It is in fact inspired by the variational principle in the original proof by Szpiro-Ullmo-Zhang. The case of concave functions f_k pertains to the situation considered in [4].

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