FLAT CURRENTS AND THEIR SLICES

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ABSTRACT. I hope this description of flat chains and their slices is less intimidating than Federer's [3], though I follow his very closely.

CONTENTS

1. CURRENTS

1.1. **Definition and basic operations.** Fix an Euclidean space $\boldsymbol{V}$ of dimension n. The metric on $\boldsymbol{V}$ induces metrics $(-,-)$ on $\Lambda^\bullet \boldsymbol{V}$ and $\Lambda^\bullet \boldsymbol{V}^*$. We will denote the corresponding norms with $|-|$.

For any open subset $\mathcal{O} \subset \boldsymbol{V}$ we denote by $\Omega^k(\mathcal{O})$ (respectively $\Omega^k_{\mathrm{cpt}}(\mathcal{O})$) the space of smooth differential k-forms on $\mathcal{O}$ (respectively smooth differential k-forms with compact support contained in $\mathcal{O}$). We denote by $\Omega_k^{\mathrm{cpt}}(\mathcal{O})$ (respectively $\Omega_k(\mathcal{O})$) their topological duals. We have a linear operator

$$\partial : \Omega_k(\mathcal{O}) \to \Omega_{k-1}(\mathcal{O}),$$

defined by

$$\langle \varphi, \partial T\rangle = \langle d\varphi, T\rangle,\ \ \forall \varphi \in \Omega^{k-1}_{\mathrm{cpt}}(\mathcal{O}).$$

Observe that any $\alpha \in \Omega^\ell(\mathcal{O})$ defines a continuous linear map

$$\alpha\cap : \Omega_k(\mathcal{O}) \to \Omega_{k-\ell}(\mathcal{O}),\ \ T \mapsto \alpha \cap T,$$

given by

$$\langle \varphi, \alpha\cap T\rangle = \langle \alpha \cup \varphi, T\rangle,\ \ \forall \varphi \in \Omega^{k-\ell}_{\mathrm{cpt}}(\mathcal{O}).$$

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Moreover,

$$(-1)^{\ell}\partial(\alpha\cap T)=\alpha\cap\partial T-d\alpha\cap T. \tag{1.1}$$

Similarly, for any smooth ℓ-vector field $\xi:\mathcal{O}\to\Lambda^{\ell}\boldsymbol{V}$ we define $\xi\wedge T\in\Omega_{k+\ell}(\mathcal{O})$ via the equality

$$\langle\varphi,\xi\wedge T\rangle=\langle\xi\lrcorner\varphi,T\rangle,\ \ \forall\varphi\in\Omega^{k+\ell}_{\rm cpt}(\mathcal{O}).$$

Suppose that $(\boldsymbol{e}_i)_{1\leq i\leq n}$ is an orthonormal basis of $\boldsymbol{V}$. We denote by $(\boldsymbol{e}^i)_{1\leq i\leq n}$ the dual orthonormal basis of $\boldsymbol{V}^*$. For any subset $I=\{1\leq i_1<\cdots<i_k\leq\}$ we set

$$\boldsymbol{e}_I=\boldsymbol{e}_{i_1}\wedge\cdots\wedge\boldsymbol{e}_{i_k},\ \ \boldsymbol{e}^I=\boldsymbol{e}^{i_1}\wedge\cdots\wedge\boldsymbol{e}^{i_k}.$$

Then the collections $(\boldsymbol{e}_I)_{|I|=k}$ and $(\boldsymbol{e}^I)_{|I|=k}$ are orthonormal bases of $\Lambda^k\boldsymbol{V}$ and respectively $\Lambda^k\boldsymbol{V}^*$. Moreover

$$T=\sum_{|I|=k}\boldsymbol{e}_I\wedge(\boldsymbol{e}^I\cap T)$$

The support of a current $T\in\Omega_k(\mathcal{O})$ is the complement of the open set

$$\big\{x\in\mathcal{O};\ \ \exists\eta\in\Omega^k_{\rm cpt}(\mathcal{O});\ \ \eta(x)\neq 0,\ \ \langle\eta,T\rangle\neq 0\,\big\}.$$

We denote by $\operatorname{supp}T$ the support of T.

Suppose that $\boldsymbol{U},\boldsymbol{V}$ are finite dimensional Euclidean spaces of dimensions m and respectively n, and $\mathcal{U}$ is an open subset of $\boldsymbol{U}$, $\mathcal{V}$ is an open subset of $\boldsymbol{V}$. For any smooth map $F:\mathcal{U}\to\mathcal{V}$ and any current $T\in\Omega_k(\mathcal{U})$ such that the restriction of F to $\operatorname{supp}T$ is proper, we define the *pushforward* $F_*T\in\Omega_k(\mathcal{V})$ by the equality

$$\langle\varphi,F_*T\rangle=\langle uF^*\varphi,T\rangle$$

where $u\in C^\infty_{\rm cpt}(\mathcal{U})$ is a function such that $u=1$ on an open neighborhood of $\operatorname{supp}T$. From the definition of the support we deduce immediately that the pushfoward is independent of the choice of cutoff function u. The resulting map $T\mapsto F_*T$ commutes with the boundary operator.

Fix orthonormal bases $(\boldsymbol{e}_i)_{1\leq i\leq m}$ and $(\boldsymbol{f}_j)_{1\leq j\leq n}$ are orthonormal bases of $\boldsymbol{U}$ and respectively $\boldsymbol{V}$. We denote by (x^i) the Euclidean coordinates determined by $(\boldsymbol{e}_i)$ and by (y^j) the Euclidean coordinates determined by $(\boldsymbol{f}_j)$.

Any $S\in\Omega_k(\mathcal{U})$ defines a linear map (called the *slant product* with S)

$$/S:\Omega^p_{\rm cpt}(\mathcal{U}\times\mathcal{V})\to\Omega^{p-k}_{\rm cpt}(\boldsymbol{V}),$$

$$\left(\sum_{|I|+|J|=p}\omega_{I,J}(x,y)dx^I\wedge dy^J\right)/S=\sum_{|J|=p-k}\left\langle\sum_{|I|=k}\omega_{I,J}dx^I,S\right\rangle dy^J,$$

If $T\in\Omega_\ell(\mathcal{V})$ the we define $S\times T\in\Omega_{k+\ell}(\mathcal{U}\times\mathcal{V})$ by the equality

$$\langle\omega,S\times T\rangle=\langle\omega/S,T\rangle,\ \ \forall\omega\in\Omega^{k+\ell}_{\rm cpt}(\mathcal{U}\times\mathcal{V}).$$

We denote by $\pi_{\boldsymbol{U}}$ (respectively $\pi_{\boldsymbol{V}}$) the natural projection $\boldsymbol{U}\times\boldsymbol{V}\to\boldsymbol{U}$ (respectively $\boldsymbol{U}\times\boldsymbol{V}\to\boldsymbol{V}$). The following simple result is often useful in proving various identities.

Proposition 1.1. *Suppose $A,B\in\Omega_p(\mathcal{U}\times\mathcal{V})$. Then $A=B$ if and only for any $\alpha\in\Omega^\bullet_{\rm cpt}(\mathcal{U})$ and $\beta\in\Omega^\bullet_{\rm cpt}(\mathcal{V})$ such that* $\deg\alpha+\deg\beta=p$ *we have*

$$\langle\pi^*_{\boldsymbol{U}}\alpha\wedge\pi^*_{\boldsymbol{V}}\beta,A\rangle=\langle\pi^*_{\boldsymbol{U}}\alpha\wedge\pi^*_{\boldsymbol{V}}\beta,B\rangle. \qquad\square$$

Here is a simple application of this principle.

Corollary 1.2. *For any $S\in\Omega_k(\mathcal{U})$ and $T\in\Omega_\ell(\mathcal{V})$ we have*

$$\partial(S\times T)=\partial S\times T+(-1)^{\dim S}S\times\partial T. \tag{1.2}$$

Proof. Let $\alpha \in \Omega^\bullet(\mathcal{U})$ and $\beta \in \Omega^\bullet(\mathcal{V})$ such that $\deg\alpha + \deg\beta = k + \ell - 1$. Then

$$\langle \pi_{\boldsymbol{U}}^*\alpha \wedge \pi_{\boldsymbol{V}}^*\beta, \partial(S \times T)\rangle = \langle d(\pi_{\boldsymbol{U}}^*\alpha \wedge \pi_{\boldsymbol{V}}^*\beta), S \times T\rangle$$

$$= \langle \pi_{\boldsymbol{U}}^* d\alpha \wedge \pi_{\boldsymbol{V}}^*\beta + (-1)^{\deg\alpha}\pi_{\boldsymbol{U}}^*\alpha \wedge \pi_{\boldsymbol{V}}^* d\beta, S \times T\rangle = \langle d\alpha, S\rangle\langle\beta, T\rangle + (-1)^{\deg\alpha}\langle\alpha, S\rangle\langle d\beta, T\rangle$$

$$= \langle\alpha, \partial S\rangle\langle\beta, T\rangle + (-1)^{\deg\alpha}\langle\alpha, S\rangle\langle\beta, \partial T\rangle = \langle\alpha, \partial S\rangle\langle\beta, T\rangle + (-1)^{\dim S}\langle\alpha, S\rangle\langle\beta, \partial T\rangle$$

$$= \langle \pi_{\boldsymbol{U}}^*\alpha \wedge \pi_{\boldsymbol{V}}^*\beta, \partial S \times T + (-1)^{\dim S} S \times \partial T\rangle.$$

□

1.2. **Currents representable by integration.** We define the *mass* of a current $T \in \Omega_k(\mathcal{O})$ to be the quantity (see also [6, Rem. 26.6])

$$\|T\| = \sup\big\{ \langle\varphi, T\rangle;\ \ T \in \Omega^k_{\rm cpt}(\mathcal{O});\ \ |\varphi(x)| \leq 1,\ \ \forall x \in \mathcal{O}\ \big\} \in [0, \infty].$$

We say that $T \in \Omega_k(\mathcal{O})$ has *locally finite mass* if $\|\eta \cap T\| < \infty$ for any $\eta \in \Omega^0_{\rm cpt}(\mathcal{O})$. Observe that this implies that for any compact subset $K \subset \mathcal{O}$ there exists a positive constant C_K such that

$$|\langle\varphi, T\rangle| \leq C \sup_{x \in K} \|\varphi(x)\|,\ \ \forall\varphi \in \Omega^k_{\rm cpt}(\mathcal{O}). \tag{1.3}$$

We have the following result.

Proposition 1.3. *Let $T \in \Omega_k(\mathcal{O})$ The following statements are equivalent.*

(a) *The current T has locally finite mass.*
(b) *The current T is representable by integration, i.e., there exists a Radon measure μ_T over U and a μ_T-measurable k-vector field $\vec{T} : \mathcal{O} \to \Lambda^k\boldsymbol{V}$ such that $|\vec{T}(x)| = 1$, μ_T- a.e. x and*

$$T = \vec{T} \wedge \mu_T,$$

i.e.

$$\langle\varphi, T\rangle = \int_{\mathcal{O}} \langle\varphi(x), \vec{T}(x)\rangle d\mu_T(x).$$

Proof. Clearly (b) ⇒ (a). The opposite implication follows from the Riesz representation theorem. Here is roughly the outline. For more details we refer to [6, §4].

Suppose that T has locally finite mass. For any open subset $U \subset \mathcal{O}$ we define

$$\tilde{\mu}_T(U) := \sup\big\{ \langle\varphi, T\rangle;\ \ \varphi \in \Omega^k_{\rm cpt}(U),\ \ \varphi(x) \leq 1,\ \ \forall x \in U\ \big\}. \tag{1.4}$$

For any $A \subset \mathcal{O}$ we set

$$\tilde{\mu}_T(A) = \inf_{U \supset A} \tilde{\mu}_T(U).$$

The correspondence $A \mapsto \tilde{\mu}_T(A)$ is an outer measure on $\mathcal{O}$ that satisfies the *Caratheodory condition*

$$\tilde{\mu}_T(A \cup B) = \tilde{\mu}_T(A) + \tilde{\mu}_T(B)\ \text{ if }\ \operatorname{dist}(A, B) > 0. \tag{1.5}$$

A subset $A \subset \mathcal{O}$ is called *measurable* if

$$\tilde{\mu}_T(S) = \tilde{\mu}_T(S \setminus A) + \tilde{\mu}_T(S \cap A),\ \ \forall S \subset \mathcal{O}.$$

The collection $\mathcal{S}_T$ of measurable subsets is a σ-algebra and we denote by μ_T the restriction of $\tilde{\mu}_T$ to $\mathcal{S}_T$. The Caratheodory condition implies that the measure μ_T is *Borel regular*, i.e.,

- $\mathcal{S}_T$ contains all the Borel sets, and
- for every $S \in \mathcal{S}_T$ there exists a Borel set $B \supset S$ such that $\mu_T(B) = \mu_T(S)$.

From the local mass condition (1.3) we deduce that μ_T satisfies the additional conditions

$$\mu_T(K) < \infty, \ \ \forall K \subset \mathcal{O} \ \text{ compact},$$

$$\mu_T(A) = \sup\bigl\{\mu_T(K);\ \ K \subset A,\ \ K \text{ compact}\bigr\}.$$

Moreover, for any nonnegative function $f \in C^0_{\mathrm{cpt}}(\mathcal{O})$ we have

$$\int_{\mathcal{O}} f(x) d\mu_T(x) = \sup\bigl\{\langle \varphi, T\rangle;\ \ |\varphi(x)| \leq f(x),\ \ \varphi \in \Omega^k_{\mathrm{cpt}}(\mathcal{O}),\ \ \forall x \in \mathcal{O}\bigr\}. \tag{1.6}$$

For any $\eta \in \Lambda^k \boldsymbol{V}^*$ we define

$$\lambda_\eta : \Omega^0_{\mathrm{cpt}}(\mathcal{O}) \to \mathbb{R},\ \ \lambda_\varphi(f) = \langle f\eta, T\rangle,\ \ \forall f \in \Omega^0_{\mathrm{cpt}}.$$

Observe that

$$|\lambda_\eta(f)| \leq |\langle f\varphi, T\rangle|$$

$$\leq \sup\bigl\{\ \langle \varphi, T\rangle;\ \ \varphi \in \Omega^k_{\mathrm{cpt}}(\mathcal{O});\ \ |\varphi(x)| \leq |f(x)| \cdot |\eta|,\ \ \forall x \in \mathcal{O}\ \bigr\} = |\eta| \int_{\mathcal{O}} |f|\, d\mu_T.$$

This implies that λ_η extends to a continuous linear functional $\lambda_\eta : L^1(\mathcal{O}, \mu_T) \to \mathbb{R}$. Thus, there exists $\nu_\eta \in L^\infty(\mathcal{O}, \mu_T)$ such that

$$\lambda_\eta(f) = \int_{\mathcal{O}} f(x)\nu_\eta(x) d\mu_T(x)\ \ \forall f \in C^0_{\mathrm{cpt}}(\mathcal{O}).$$

Note that $\|\nu_\eta\|_{L^\infty} \leq |\eta|$. Now fix a basis $(\boldsymbol{e}_I)$ of $\Lambda^k\boldsymbol{V}$, denote by $(\boldsymbol{e}^I)$ the dual basis of $\Lambda^k\boldsymbol{V}^*$ define $\xi : \mathcal{O} \to \Lambda^k\boldsymbol{V}$ via the equality

$$\vec{T}(x) = \sum_I \xi_I(x)\boldsymbol{e}_I,\ \ \xi_I(x)\nu_{\boldsymbol{e}^I}(x).$$

We deduce that $|\vec{T}(x)| \leq 1$, and

$$\langle \eta, T\rangle = \int_{\mathcal{O}} \langle \varphi(x), \vec{T}(x)\rangle d\mu_T(x),\ \ \forall \varphi \in \Omega^k_{\mathrm{cpt}}(\mathcal{O}).$$

The equality $|\vec{T}(x)| = 1$ is proved observing that for a countable, dense, open subset $\mathcal{F} \subset \Lambda^k\boldsymbol{V}^*$ we have

$$\langle \eta, \vec{T}(x)\rangle = \lim_{\varepsilon \searrow 0} \frac{1}{\mu_T(B(x,\varepsilon))} \int_{B(x,\varepsilon)} \langle \eta, \vec{T}(y)\rangle d\mu_T(y),\ \ \forall \eta \in \mathcal{F},\ \ \text{a.e. } x \in \mathcal{O}.$$

□

If $T \in \Omega_k(\mathcal{O})$ is representable by integration, then the map

$$\Omega^k_{\mathrm{cpt}}(\mathcal{O}) \ni \varphi \mapsto \langle \varphi, T\rangle \in \mathbb{R}$$

extends by $L^1(\mathcal{O}, \mu_T)$ continuity to a linear map on the space of bounded, compactly supported, Borel measurable k-forms on $\mathcal{O}$. In particular, if η is a bounded, Borel measurable ℓ-form on $\mathcal{O}$, we can define the $(k-\ell)$-current $\eta \cap T$. Note that if $B \subset \mathcal{O}$ is a Borel set, then the characteristic function $\mathbb{1}_B$ is a bounded, Borel measurable 0-form and we define the *restriction of T to B* to be the current

$$T|_B := \mathbb{1}_B \cap T. \tag{1.7}$$

Example 1.4. Suppose that M is a compact, orientable m-dimensional C^1-submanifold of $\boldsymbol{V}$. Then any orientation $\boldsymbol{or}$ on M defines a current $[M, \boldsymbol{or}] \in \Omega_m(\boldsymbol{V})$,

$$\langle \omega, [M, \boldsymbol{or}] \rangle = \int_{(M, \boldsymbol{or})} \omega, \;\; \forall \omega \in \Omega^m_{\mathrm{cpt}}(\boldsymbol{V}).$$

The orientation $\boldsymbol{or}$ defines a continuous, unit length section $\xi_{M,\boldsymbol{or}}$ of $\Lambda^m TM$. The current $[M, \boldsymbol{or}]$ is reprsentable by integration

$$[M, \boldsymbol{or}] = \xi_{M,\boldsymbol{or}} \wedge d\mathcal{H}^m,$$

where $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure. Moreover

$$\|\, [M, \boldsymbol{or}] \,\| = \mathcal{H}^m\big(M \big). \qquad \square$$

Let us observe that if $\boldsymbol{U}$ and $\boldsymbol{V}$ are finite dimensional Euclidean spaces, $\mathcal{U} \subset \boldsymbol{U}$, $\mathcal{V} \subset \boldsymbol{V}$ are open subsets, $S \in \Omega_k(\mathcal{U})$, $T \in \Omega_\ell(\mathcal{V})$ are currents representable by integration, then $S \times T$ is representable by integration and

$$\mu_{S\times T} = \mu_S \times \mu_T, \; ; \overrightarrow{S \times T} = \vec{S} \wedge \vec{T}.$$

1.3. **Locally normal and locally flat currents.** A current $T \in \Omega_k(\mathcal{O})$ is called *normal* if it has *compact support* and

$$\boldsymbol{N}(T) := \|T\| + \|\partial T\| < \infty.$$

We denote by $\boldsymbol{N}_k(\mathcal{O})$ the space of normal k-dimensional currents.

We say that $T \in \Omega_k(\mathcal{O})$ is *locally normal* if $f \cap T$ is normal for any $f \in \Omega^0_{\mathrm{cpt}}(\mathcal{O})$. Note that T is locally normal iff both T and ∂T are representable by integration. We denote by $\boldsymbol{N}^{\mathrm{loc}}_k(\mathcal{O})$ the vector space of locally normal currents.

For any compact subset $K \subset \mathcal{O}$ and any $\varphi \in \Omega^\ell(\mathcal{O})$ we set

$$\|\varphi\|_K := \sup_{x \in K} \|\varphi(x)\|,$$

and we define the *flat seminorm*

$$\boldsymbol{F}_K(\varphi) := \max\big\{ \|\varphi\|_K, \; \|d\varphi\|_K \,\big\}.$$

For $T \in \Omega_k(\mathcal{O})$ we define the dual *flat seminorm*

$$\boldsymbol{F}_K(T) = \sup\big\{ \, \langle \varphi, T \rangle; \;\; \boldsymbol{F}_K(\varphi) \leq 1 \,\big\}.$$

Let us observe that

$$\boldsymbol{F}_K(T) < \infty \Rightarrow \operatorname{supp} T \subset K.$$

Proposition 1.5. *Let $T \in \Omega_\ell(\mathcal{O})$ and K a compact subset of $\mathcal{O}$. If* $\operatorname{supp} T \subset K$ *then*

$$\boldsymbol{F}_K(T) = \inf\big\{ \|T - \partial S\| + \|S\|; \;\; S \in \Omega_{k+1}(\mathcal{O}), \;\; \operatorname{supp} S \subset K \,\big\}.$$

Proof. Suppose that $S \in \Omega_{k+1}(\mathcal{O})$, $\operatorname{supp} S \subset K$. Then for any $\varphi \in \Omega^k_{\mathrm{cpt}}(\mathcal{O})$ such that $\|\varphi\|_K \leq 1$ we have

$$\langle \varphi, T \rangle = \langle \varphi, T - \partial S \rangle + \langle d\varphi, S \rangle \leq \|T - \partial S\| + \|S\|.$$

This proves that

$$\boldsymbol{F}_K(T) \leq \inf\big\{ \|T - \partial S\| + \|S\|; \;\; S \in \Omega_{k+1}(\mathcal{O}), \;\; \operatorname{supp} S \subset K \,\big\}.$$

The equality follows from the following *key existence result*.

Lemma 1.6. *If $T \in \Omega_\ell(\mathcal{O})$, and $\boldsymbol{F}_K(T) < \infty$, then there exist $R \in \Omega_\ell(\mathcal{O})$, $S \in \Omega_{\ell+1}(\mathcal{O})$ such that*

$$\begin{aligned} \operatorname{supp} R, \operatorname{supp} S \subset K,\\ T = R + \partial S,\\ \boldsymbol{F}_K(T) = \|R\| + \|S\|. \end{aligned}$$

□

The proof is a direct application of the Hahn-Banach theorem. In particular, it is *nonconstructive*. For details we refer to [3, §4.1.12]. □

We denote by $\boldsymbol{F}_{\ell,K}(\mathcal{O})$ the closure with respect to the seminorm $\boldsymbol{F}_K$ of the space

$$\boldsymbol{N}_{\ell,K}(\mathcal{O}) := \bigl\{ T \in \Omega_\ell(\mathcal{O});\ \ \operatorname{supp} T \subset K,\ \ \boldsymbol{N}(T) < \infty \bigr\}.$$

and we set

$$\boldsymbol{F}_\ell(\mathcal{O}) = \bigcup_{K \subset \mathcal{O}} \boldsymbol{F}_{\ell,K}(\mathcal{O}).$$

We will refer to the currents in $\boldsymbol{F}_\ell(\mathcal{O})$ as *flat currents*. Observe that, by definition, the flat currents have *compact support*.

A current $T \in \Omega_\ell(\mathcal{O})$ is called *locally flat* if for any $f \in C^\infty_{\mathrm{cpt}}(\mathcal{O})$ the current $fT = f \cap T \in \Omega_\ell^{\mathrm{cpt}}(\mathcal{O})$ is flat. We denote by $\boldsymbol{F}_\ell^{\mathrm{loc}}(\mathcal{O})$ the vector space of locally flat currents. Observe that

$$\boldsymbol{N}_\ell^{\mathrm{loc}}(\mathcal{O}) \subset \boldsymbol{F}_\ell^{\mathrm{loc}}(\mathcal{O})$$

and moreover

$$\partial \boldsymbol{N}_\ell^{\mathrm{loc}}(\mathcal{O}) \subset \boldsymbol{N}_{\ell-1}^{\mathrm{loc}}(\mathcal{O}),\ \ \partial \boldsymbol{F}_\ell^{\mathrm{loc}}(\mathcal{O}) \subset \boldsymbol{F}_{\ell-1}^{\mathrm{loc}}(\mathcal{O}).$$

Suppose that $\boldsymbol{U}$ and $\boldsymbol{V}$ are finite dimensional Euclidean vector spaces, $\mathcal{U} \subset \boldsymbol{U}$ is an open set, $F : \mathcal{U} \to \boldsymbol{V}$ is a smooth map, and $T \in \Omega_k(\mathcal{U})$. If T is representable by integration then $F_*(T)$ is representable by integration and

$$\mu_{F_*T} \leq F_*\bigl(\|F_*\vec{T}\|\mu_T \bigr), \tag{1.8}$$

where $\|F_*\vec{T}|$ denotes the measurable function

$$\mathcal{O} \ni x \mapsto \bigl\| D_x F\bigl(\vec{T}(x) \bigr) \bigr\| \in [0, \infty).$$

This shows that if $T \in \boldsymbol{N}_\ell(\mathcal{U})$ (resp. $\boldsymbol{F}_\ell(\mathcal{U})$) then $F_*(T) \in \boldsymbol{N}_\ell(\boldsymbol{V})$ (resp. $F_*(T) \in \boldsymbol{F}_\ell(\boldsymbol{V})$).

1.4. **Homotopies.** Suppose $\mathcal{U}$ (resp. $\mathcal{V}$) is an open subset of the Euclidean spaces $\boldsymbol{U}$ (resp. $\boldsymbol{V}$) and

$$H : [0, 1] \times \mathcal{U} \to \boldsymbol{V},$$

is a smooth map. We denote by H_t the restriction of H to the slices $\{t\} \times \mathcal{U}$. Let $[\![0, 1]\!] \in \Omega_1(\mathbb{R})$ denote the current of integration over $[0, 1]$ equipped with its natural orientation. Observe that for any $T \in \Omega_k(\mathcal{U})$ we have

$$\partial H_*([\![0, 1]\!] \times T) = H_*\bigl(\partial [\![0, 1]\!] \times T - [\![0, 1]\!] \times \partial T\bigr)$$

so that

$$(H_1)_*T - (H_0)_*T = \partial H_*([\![0, 1]\!] \times T) + H_*\bigl([\![0, 1]\!] \times \partial T\bigr). \tag{1.9}$$

Using the inequality (1.8) we deduce that if T is reprsentable by integration than $H_*([\![0, 1]\!] \times T)$ is representable by integration and for anu open subset $\mathcal{O} \subset \mathcal{V}$ we have

$$\mu_{H_*([\![0,1]\!] \times T)}(\mathcal{O}) \leq \int_0^1 \left(\int_{H_t^{-1}(\mathcal{O})} |\dot{H}_t(x) \wedge DH_t(\vec{T}(x))| d\mu_T(x) \right) dt. \tag{1.10}$$

In the remainder of this subsection we assume that If H is an affine homotopy

$$H_t = (1-t)H_0 + tH_1,$$

and we set

$$\rho(x) := \max\big\{ \|DH_0(x)\|,\ \|DH_1(x)\|\,\big\}.$$

We deduce

$$\mu_{H_*([\![0,1]\!]\times T}(\mathcal{O}) \leq \int_{H^{-1}(\mathcal{O})} |H_1(x) - H_0(x)|\rho(x)^k d\mu_T(x), \tag{1.11}$$

and

$$\big\|\, H_*([\![0,1]\!]\times T)\,\big\| \leq \sup_{x\in\mathcal{U}} |H_1(x) - H_0(x)| \times \sup_{x\in\mathcal{U}} \rho(x)^k \times \|T\|. \tag{1.12}$$

Suppose now that T is normal. In particular, it has compact support, and we define

$$C := H\big([0,1]\times \operatorname{supp} T\big),\ \ S := H_*([\![0,1]\!]\times T).$$

Using (1.9) we deduce

$$(H_1)_*T - (H_0)_*T - \partial S = H_*([\![0,1]\!]\times \partial T).$$

Invoking Proposition 1.5 we conclude

$$\begin{aligned} \boldsymbol{F}_{k,C}\big(\,(H_1)_*T - (H_0)_*T\,\big) &\leq \big\|H_*([\![0,1]\!]\times\partial T)\,\big\| + \big\|\,H_*([\![0,1]\!]\times T)\,\big\| \\ &\stackrel{(1.12)}{\leq} \|H_1 - H_0\|_{L^\infty(\operatorname{supp} T)}\Big(\|T\|\cdot\|\rho\|^k_{L^\infty(\operatorname{supp} T)} + \|\partial T\|\cdot\|\rho\|^{k-1}_{L^\infty(\operatorname{supp} T)}\Big). \end{aligned} \tag{1.13}$$

1.5. **Lipschitzian pushfoward.** Suppose $\mathcal{U}$ (resp. $\mathcal{V}$) is an open subset of the Euclidean space $\boldsymbol{U}$ (resp $\boldsymbol{V}$), $K\subset\mathcal{U}$ is a compact subset. We assume that $\mathcal{V}$ is a *convex* set and $F:\mathcal{U}\to\mathcal{V}$ is a locally Lipschitzian map.

For any smooth maps $H_0, H_1:\mathcal{U}\to\mathcal{V}$ we denote by $C(H_0,H_1)$ the convex hull of $H_0(K)\cup H_1(K)$, by L_{H_i} the Lipschitz constant of the restriction of H_i to K, and we set

$$L_{H_0,H_1} := \max\big\{\,L_{H_0}, L_{H_1}\,\big\}.$$

From (1.13) we deduce that if $T\in\boldsymbol{N}_{m,K}(\mathcal{U})$, then

$$\boldsymbol{F}_{C(H_0,H_1)}\big((H_1)_*T - (H_0)_*T\,\big) \leq \|H_1 - H_0\|_{L^\infty(K)}\big(\|T\|L^m_{H_0,H_1} + \|\partial T\|L^{m-1}_{H_0,H_1}\,\big). \tag{1.14}$$

Suppose that $F_n:\mathcal{U}\to\mathcal{V}$ is a sequence of smooth maps with the following properties.

(a) The sequence converges uniformly to F on K.
(b) The sequence L_{F_n} is bounded.

For any compact neighborood C of $F(K)$ there exists $n=n(C)$ such that

$$(F_n)_*T\in\boldsymbol{N}_{m,\mathcal{N}}(\mathcal{V}),\ \ \forall n\geq n(C),$$

and the sequence $(F_n)_*T\in\boldsymbol{N}_{m,\mathcal{N}}(\mathcal{V})$, $n\geq n(C)$ is Cauchy in the $\boldsymbol{F}_{\mathcal{N}}$-metric. This is a complete metric so this sequence is convergent in this metric. The limit current is supported on $F(K)$. The inequality (1.14) also shows that the limit is independent of the choice of smooth map F_n with the above properties. We define the pushforward F_*T to be this common limit. In other words, we have succeeded in giving an unambiguous meaning of the pushforward of a normal current by a locally Lipschitz map. We get in this fashion a linear map

$$F_*:\boldsymbol{N}_{m,K}(\mathcal{U})\to\boldsymbol{N}_{m,F(K)}(\mathcal{V}).$$

Observe that

$$\|F_*T\|\leq L^m_F\|T\|. \tag{1.15}$$

From Proposition 1.5 we deduce that there exists a current $S \in \Omega_{m+1}(\mathcal{U})$ such that $\operatorname{supp} S \subset K$ and

$$\boldsymbol{F}_K(T) = \|T - \partial S\| + \|S\|.$$

Since T has finite mass we deduce that ∂S has finite mass and thus S is normal. We deduce

$$\boldsymbol{F}_{F(K)}\big(\, F_* T\,\big) \leq \|F_*(T - \partial S)\| + \|F_* S\| \leq L_F^m \|T - \partial S\| + L_F^{m+1}\|S\|$$
$$\leq \max(L_F, 1) L_F^m \boldsymbol{F}_K(T).$$

Since by definition $\boldsymbol{N}_{m,K}(\mathcal{U})$ is $\boldsymbol{F}_K$-dense in $\boldsymbol{F}_{m,k}(\mathcal{U})$ we deduce from the above inequality that the push-forward extends by continuity to a linear map

$$F_* : \boldsymbol{F}_{m,K}(\mathcal{U}) \to \boldsymbol{F}_{m,F(K)}(\mathcal{V}),$$

satisfying the bound

$$\boldsymbol{F}_{F(K)}\big(\, F_* T\,\big) \leq \max(L_F, 1) L_F^m \boldsymbol{F}_K(T), \ \ \forall T \in \boldsymbol{F}_{m,K}(\mathcal{U}) \tag{1.16}$$

The above considerations lead immediately to the following conclusion.

Corollary 1.7. *If $F_n : \mathcal{U} \to \mathcal{V}$ is a sequence of smooth Lipschitz maps satisfying the conditions (a) and (b) above then for any compact neighborhood C of $F(K)$ in $\mathcal{V}$ we have*

$$\lim_{n\to\infty} \boldsymbol{F}_C((F_n)_* T - F_* T) \to 0, \ \ \forall T \in \boldsymbol{F}_{m,K}(\mathcal{U}). \qquad \square$$

This is a nontrivial result even when F is C^1 because above we *do not require* C^1 convergence $F_n \to F$.

1.6. **Properties of flat currents.** Corollary 1.7 has nontrivial consequences. We want to discuss one of them here.

Proposition 1.8. *Suppose that $\boldsymbol{V}$ is an Euclidean space of dimension n and $T \in \boldsymbol{F}_k(\boldsymbol{V})$. If $\boldsymbol{U}$ is another finite dimensional Euclidean space $\mathcal{O}$ is an open neighborhood of $\operatorname{supp} T$ and $F, G : \mathcal{O} \to \boldsymbol{U}$ are locally Lipschitz maps such $F|_{\operatorname{supp} T} = G|_{\operatorname{supp} T}$, then $F_* T = G_* T$.*

Proof. For $r > 0$ define $\Psi_r : \boldsymbol{U} \to \boldsymbol{U}$ by the equality

$$\Psi_r(\boldsymbol{u}) = \begin{cases} 0, & |\boldsymbol{u}| \leq r \\ \big(1 - \frac{r}{|\boldsymbol{u}|}\big) v, & |\boldsymbol{u}| > r. \end{cases}$$

The map Ψ_r is Lipschitz with Lipschitz constant ≤ 1 and

$$\big|\Psi_r(\boldsymbol{u}) - \boldsymbol{u}\big| \leq r, \ \ \forall \boldsymbol{u} \in \boldsymbol{U}. \tag{1.17}$$

We fix a smooth, nonnegative, function $\Phi : \boldsymbol{V} \to \mathbb{R}$ such that

$$\int_{\boldsymbol{U}} \Phi(\boldsymbol{u}) |d\boldsymbol{u}| = 1 \ \text{ and } \ \Phi(\boldsymbol{u}) = 0, \ \ \forall |\boldsymbol{u}| \geq 1.$$

We set

$$\Phi_\varepsilon(\boldsymbol{u}) := \frac{1}{\varepsilon^n} \Phi\left(\frac{\boldsymbol{u}}{\varepsilon}\right),$$

so that $(\Phi_\varepsilon)_{\varepsilon > 0}$ is a mollifying family. We define

$$G_r(\boldsymbol{v}) = F(\boldsymbol{v}) + \Psi_r\big(G(\boldsymbol{v}) - F(\boldsymbol{v})\,\big).$$

From (1.17) we deduce that

$$\big|G_r(\boldsymbol{v}) - G(\boldsymbol{v})| \leq r, \ \ \forall \boldsymbol{v} \in \mathcal{O},$$

and

$$G_r(\boldsymbol{v}) = F(\boldsymbol{v}) \ \text{ if } \ |F(\boldsymbol{v}) - G(\boldsymbol{v})| \leq r.$$

Observe that the maps $\Phi_\varepsilon * F$ and $\Phi_\varepsilon * G_r$ coincide on the set

$$\mathcal{O}_{r,\varepsilon} := \big\{ \boldsymbol{v} \in \mathcal{O};\ \ |F(\boldsymbol{v}') - G(\boldsymbol{v}')| < r,\ \ \forall \boldsymbol{v}' \in B(\boldsymbol{v}, \varepsilon) \big\}.$$

For $r, \varepsilon > 0$ sufficiently small $\mathcal{O}_{r,\varepsilon}$ is a neighborhood of $\operatorname{supp} T$. Moreover, the maps $\Phi_\varepsilon * F$ and $\Phi_\varepsilon * G_r$ approximate F and respectively G on any compact $K \subset \mathcal{U}$. The proposition now follows from Corollary 1.7. □

Remark 1.9. The above proposition shows that the pushfoward of a flat current by a locally Lipschitz map is oblivious to the infinitesimal neighborhood of the support of the current. Consider for example the current representable by integration

$$T = \partial_x \wedge \delta_0 \in \Omega_1(\mathbb{R}),$$

where δ_0 is the Dirac measure concentrated at the origin. Then $\operatorname{supp} T = \{0\}$ The maps

$$F, G : \mathbb{R} \to \mathbb{R},\ \ F(x) = 0,\ \ G(x) = x,\ \ \forall x \in \mathbb{R}$$

coincide at the origin. However, $F_* T = 0$ and $G_* T = T$. □

Corollary 1.10. *Suppose $\boldsymbol{V}$ is a finite dimensional Euclidean space, $\boldsymbol{U}$ is a subspace of $\boldsymbol{V}$ and T is a flat currrent with support contained in $\boldsymbol{U}$. If $\dim T > \dim \boldsymbol{U}$, then $T = 0$.*

Proof. Denote by $P_{\boldsymbol{U}} : \boldsymbol{V} \to \boldsymbol{U}$ the orthogonal projection onto $\boldsymbol{U}$ and by $I_{\boldsymbol{U}} : \boldsymbol{U} \to \boldsymbol{V}$ the canonical inclusion.

Then the maps $\mathbb{1}_{\boldsymbol{V}}$ and $I_{\boldsymbol{U}} \circ P_{\boldsymbol{U}}$ coincide on $\boldsymbol{U}$ and thus on the support of T. Hence

$$T = (\mathbb{1}_{\boldsymbol{V}})_* T = (P_{\boldsymbol{U}})_* (I_{\boldsymbol{U}})_* T.$$

Now observe that since $\dim T > \dim \boldsymbol{U}$ the current $(I_{\boldsymbol{U}})_* T \in \Omega_{\dim T}(\boldsymbol{U})$ is trivial. □

Suppose that $\boldsymbol{V}$ is an Euclidean vector space of dimension n. For every $0 \leq m \leq n$ we denote by $\mathcal{X}_m$ the space of Lebesgue integrable, compactly supported maps

$$\xi : \boldsymbol{V} \to \Lambda^m \boldsymbol{V}.$$

To every pair $(\xi, \eta) \in \mathcal{X}_m \times \mathcal{X}_{m+1}$ we associate the compactly supported current

$$\mathcal{T}_{\xi,\eta} = \xi \wedge d\mathcal{H}^n_{\boldsymbol{V}} + \partial(\eta \wedge d\mathcal{H}^n_{\boldsymbol{V}}),$$

where $d\mathcal{H}^n_{\boldsymbol{V}}$ is the usual Lebesgue measure on $\boldsymbol{V}$. Observe that

$$\operatorname{supp} \mathcal{T}_{\xi,\eta} \subset \operatorname{supp} \xi \cup \operatorname{supp} \eta =: \operatorname{supp}(\xi, \eta).$$

Moreover, for any compact $K \supset \operatorname{supp}(\xi, \eta)$ and any $\varphi \in \Omega^m_{\text{cpt}}(\boldsymbol{V})$ such that

$$\boldsymbol{F}_K(\varphi) = \sup_{x \in K} \max\big\{ |\varphi(x)|,\ |d\varphi(x)| \leq 1 \big\} \leq 1,$$

we have

$$\langle \varphi, \mathcal{T}_{\xi,\eta} \rangle = \int_{\boldsymbol{V}} \langle \varphi(x), \xi(x) \rangle d\mathcal{H}^n(x) + \int_{\boldsymbol{V}} \langle d\varphi(x), \eta(x) \rangle d\mathcal{H}_n(x) \leq \|\xi\|_{L^1} + \|\eta\|_{L^1}.$$

This proves that $\mathcal{T}_{\xi,\eta}$ is flat, $\mathcal{T}_{\xi,\eta} \in \boldsymbol{F}_{m,K}(\boldsymbol{V})$, and

$$\boldsymbol{F}_K(T) \leq \|\xi\|_{L^1} + \|\eta\|_{L^1} = \|\xi \wedge d\mathcal{H}^n_{\boldsymbol{V}}\| + \|\eta \wedge d\mathcal{H}^n_{\boldsymbol{V}}\|.$$

The next result essentially states that all flat currents are of the form $\mathcal{T}_{\xi,\eta}$.

Proposition 1.11. *Suppose that K is a compact subset and $T \in \boldsymbol{F}_{m,K}(\boldsymbol{V})$. For any $r > 0$ we set*

$$K_r := \big\{ \boldsymbol{v} \in \boldsymbol{V};\ \ \mathrm{dist}(\boldsymbol{v}, K) \leq r \big\}.$$

Then for any $\delta > 0$ there exist $(\xi_\delta, \eta_\delta) \in \mathcal{X}_m \times \mathcal{X}_{m+1}$ such that

$$\begin{aligned} \mathrm{supp}\,\xi_\delta \cup \mathrm{supp}\,\eta_\delta \subset K_\delta, \\ T = \mathcal{T}_{\xi_\delta, \eta_\delta}, \\ \|\xi_\delta\|_{L^1} + \|\eta_\delta\|_{L^1} \leq \boldsymbol{F}_K(T) + \delta. \end{aligned}$$

The proof of this result is via a decreasing induction on m aided by Lemma 1.6. More precisely on writes T as a an infinite sum

$$T = \sum_{j=0}^{\infty} (R_j + \partial S_j)$$

convergent in the flat norm, where for any j

$$R_k \in \boldsymbol{N}_m(\boldsymbol{V}),\ \ S_j \in \boldsymbol{N}_{m+1}(\boldsymbol{V}),\ \ \ \mathrm{supp}\, R_j, K_k \subset K_{2^{(-j+3)}\delta}$$

and R_j is smooth. Because R_j is smooth we can write

$$R_j = \xi_j \wedge d\mathcal{H}^n,\ \ \xi_j \in C^\infty(\boldsymbol{V}, \Lambda^m \boldsymbol{V}).$$

By induction we can write

$$S_j = \eta_j \wedge d\mathcal{H}^n + \partial(\zeta_j \wedge d\mathcal{H}^n),\ \ \xi_j \in L^1(\boldsymbol{V}, \Lambda^{m+1}\boldsymbol{V}),\ \ \zeta_j \in L^1(\boldsymbol{V}, \Lambda^{m+1}\boldsymbol{V})$$

and one can can show that

$$\sum_j (\|\xi_j\|_{L^1} + \eta_j\|_{L^1}) < \infty.$$

For details we refer to [3, §4.1.18]. In the next example we explain the construction of ξ_δ and η_δ in some special but illuminating cases.

Example 1.12. (a) Suppose $T \in \Omega_0(\mathbb{R})$ is given by the Dirac measure supported at the origin. The equality $T = \mathcal{T}_{\xi,\eta}$ signifies that η is a compactly supported intgerable function on $\mathbb{R}$, η is a copactly supported L^1-vector field on $\mathbb{R}$ such that

$$f(0) = \int_{\mathbb{R}} \Big(f(x)\xi(x) + \frac{df}{d\eta}(x) \Big) dx,\ \ \forall f \in C^\infty_{\mathrm{cpt}}(\mathbb{R}).$$

We can represent $\eta(x)$ in the form $w(x)\frac{d}{dx}$, $w \in L^1$ and we can rewrite the above equality as

$$f(0) = \int_{\mathbb{R}} f(x)\xi(x)dx + \int_{\mathbb{R}} w(x)\frac{df}{dx}(x)dx,\ \ \forall f \in C^\infty_{\mathrm{cpt}}(\mathbb{R}),$$

or as an equality of distributions

$$\frac{dw}{dx} = -\delta_0 + \xi(x).$$

We seek compactly supported L^1-solutions $(\xi(x), w(x))$ of the above equation.

Fix a smooth, function $\Phi : \mathbb{R} \to [0, \infty)$ with support on $[-1, 1]$ such that

$$\int_{\mathbb{R}} \Phi(x)dx = 1.$$

For $r > 0$ we set

$$\Phi_r(x) = \frac{1}{r}\Phi\left(\frac{x}{r}\right).$$

The measure $\Phi_r(x)|dx|$ is a 0-current that converges to δ_0. Define

$$w_{r,\varepsilon}(x) = \int_{-\infty}^x \Big(\Phi_r(t) - \Phi_\varepsilon(t) \Big) dt.$$

Observe that $\operatorname{supp} w_{r,\varepsilon} \subset [-r, r]$ and

$$\frac{dw_{r,\varepsilon}}{dx} = \Phi_r - \Phi_\varepsilon.$$

It is easy to check that $w_{r,\varepsilon}$ converges as $\varepsilon \to 0$ to a L^1-function supported on $[-r, r]$ and satisfying the distributional equation

$$\frac{dw_r}{dx} = -\delta_0 + \Phi_r(x).$$

(b) For any $\xi \in \Lambda^m \boldsymbol{V}$ we denote by $\xi_\dagger \in \Lambda^m \boldsymbol{V}^*$, i.e.,

$$(\eta, \xi_\dagger) = \langle \eta, \xi \rangle, \quad \forall \eta \in \Lambda^m \boldsymbol{V}^*.$$

Fix an orientation $\boldsymbol{or}$ on $\boldsymbol{V}$, denote by $\Omega_{\boldsymbol{V}} \in \Omega^n(\boldsymbol{V})$ the metric volume defined by this orientation and by $*$ the Hodge star operator

$$* : \Omega^k(\boldsymbol{V}) \to \Omega^{n-k}(\boldsymbol{V}^*).$$

If $\xi \in \mathfrak{X}_m$ then for any $\varphi \in \Omega^m_{\mathrm{cpt}}(\boldsymbol{V})$ we have

$$\langle \varphi, \xi \wedge d\mathcal{H}^n_{\boldsymbol{V}} \rangle = \int_{\boldsymbol{V}} \langle \varphi, \xi \rangle \Omega_{\boldsymbol{V}} = \int_{\boldsymbol{V}} (\varphi, \xi_\dagger) \Omega_{\boldsymbol{V}} = \int_{\boldsymbol{V}} \varphi \wedge *\xi_\dagger$$

$$= (-1)^{m(n-m)} \int_{\boldsymbol{V}} *\xi_\dagger \wedge \varphi = (-1)^{m(n-m)} \big\langle\, \varphi, *\xi_\dagger \cap [\boldsymbol{V}, \boldsymbol{or}] \,\big\rangle$$

Hence

$$\xi \wedge d\mathcal{H}^n_{\boldsymbol{V}} = (-1)^{m(n-m)} * \xi_\dagger \cap [\boldsymbol{V}, \boldsymbol{or}].$$

Using (1.1) we deduce

$$(-1)^{n-m+m(n-m)} \partial(\xi \wedge d\mathcal{H}^n) = -d * \xi_\dagger \cap [\boldsymbol{V}, \boldsymbol{or}].$$

We set

$$\chi(n, m) := 1 + (n - m) + m(n - m) \bmod 2,$$

and we deduce

$$\partial(\xi \wedge d\mathcal{H}^n) = (-1)^{\chi(n,m)} (d * \xi_\dagger) \cap [\boldsymbol{V}, \boldsymbol{or}]$$

The equality $T = \mathcal{T}_{\xi,\eta}$ becomes

$$T = \big((-1)^{m(n-m)} * \xi_\dagger + (-1)^{\chi(n,m+1)} d * \eta_\dagger \big) \cap [\boldsymbol{V}, \boldsymbol{or}] =: (\tau + d\sigma) \cap [\boldsymbol{V}, \boldsymbol{or}],$$

$\tau \in \Omega^{n-m}_{\mathrm{cpt}}(\boldsymbol{V})$, $\sigma \in \Omega^{n-m-1}_{\mathrm{cpt}}(\boldsymbol{V})$. If T is the current of integration defined by a smooth compact oriented submanifold of $\boldsymbol{V}$ without boundary then τ would be a Thom form of the normal bundle and σ would be an angular form of the punctured normal bundle, [1]. □

Corollary 1.13. *Suppose that K is a compact subset of $\mathbb{R}^n$ and $T \in \boldsymbol{F}_{n,K}(\mathbb{R}^n)$. Then T has finite mass and the measure μ_T is absolutely continuous with respect to $\mathcal{H}^n$. If we set*

$$\rho_T := \frac{d\mu_T}{d\mathcal{H}^n} \in L^1(\mathbb{R}^n, d\mathcal{H}^n),$$

then

$$d\mu_T(x) = \rho_T(x) d\mathcal{H}^n(x)$$

and

$$T=\xi_T(x)\wedge d\mathcal{H}^n,\;\;\xi_T(x)=\rho_T(x)\vec{T}(x).\qquad\square$$

We say that a flat current $T\in \boldsymbol{F}_\ell(\mathcal{O})$, $\mathcal{O}$ open in $\boldsymbol{V}$, is smooth if it can be written as $T=\mathcal{T}_{\xi,\eta}$ ξ,η smooth. The space of smooth flat currents with support in a compact set is clearly dense with respect to the $\boldsymbol{F}_{\ell,K_r}$-norm, r very small, where

$$K_r:=\big\{\,\boldsymbol{v};\;\operatorname{dist}(\boldsymbol{v},K)\leq r\,\big\}.$$

For r sufficiently small we have $K_r\subset\mathcal{O}$. Any current $T\in\boldsymbol{F}_{\ell,K}(\mathcal{O})$ can be approximated in $\boldsymbol{F}_{K_r}$-norm b smooth flat currents. Indeed using molllifiers, we obtain a family of smooth flat currents $T_\varepsilon\in\boldsymbol{F}_{\ell,K_\varepsilon}(\mathcal{O})$ such that

$$\boldsymbol{F}_{K_r}(T_\varepsilon-T)\to 0\;\;\text{as}\;\;\varepsilon\to 0.$$

Suppose now that K admits a compact neighborhood $\mathcal{N}\subset\mathcal{O}$ such that there exists a Lipschitz retraction $r:\mathcal{N}\to K$. Then $r_*(T)=0$ and

$$\boldsymbol{F}_{K_r}\big(\,r_*T_\varepsilon-r_*T\big)\to 0\;\;\text{as}\;\;\varepsilon\to 0.$$

We have thus proved the following result.

Corollary 1.14. *If K is a Lipschitz neighborhood retract in $\mathcal{O}$ then the space $\boldsymbol{F}_{\ell,K}(\mathcal{O})$ is separable with respect to the $\boldsymbol{F}_K$ norm.* $\square$

From Proposition 1.11 and mollifiers we deduce the following refinement of Lemma 1.6.

Proposition 1.15. *If $\mathcal{V}$ is an open subset of the n-dimensional Euclidean space $\boldsymbol{V}$, K is a compact subset of $\mathcal{V}$ and $T\in\boldsymbol{F}_{\ell,K}(\mathcal{V})$ then there exist $R\in\boldsymbol{F}_{\ell,K}(\mathcal{V})$ and $S\in\boldsymbol{F}_{\ell+1,K}(\mathcal{V})$ such that*

$$T=R+\partial S,\;\;\boldsymbol{F}_K(T)=\|R\|+\|S\|.\qquad\square$$

2. SLICING

2.1. Notations.

- We denote by $\boldsymbol{U}$an oriented Euclidean space of dimension N.
- We denote by $\mathcal{U}$ an open subset of $\boldsymbol{U}$.
- We denote by $F:\mathcal{U}\to\mathbb{R}^n$ a locally Lipschitz map.
- We denote by $\Omega_{\boldsymbol{U}}$ the canonical volume form on $\boldsymbol{U}$ determined by the Euclidean metric and the orientation.
- We denote by $(x^1,\dots,x^n)$ the canonical Euclidean coordinates on $\mathbb{R}^n$, and by $\boldsymbol{\omega}_n$ the volume of the unit ball in $\mathbb{R}^n$.
- For $r>0$ and $\boldsymbol{y}\in\mathbb{R}^n$ we set

$$\Omega_{\boldsymbol{y},r}:=\frac{1}{\boldsymbol{\omega}_n r^n}\mathbb{1}_{B(\boldsymbol{y},r)}\Omega,\;\;\Omega:=dx^1\wedge\cdots\wedge dx^n,$$

  where $\mathbb{1}_S$ denotes the characteristic function of a set $S\subset\mathbb{R}^n$.

Observe that $\Omega_{\boldsymbol{y},r}$ defines a 0-current $\delta_{\boldsymbol{y},r}\in\Omega_0(\mathbb{R}^n)$

$$\langle f,\delta_{\boldsymbol{y},r}\rangle=\frac{1}{\boldsymbol{\omega}_n r^n}\int_{B(\boldsymbol{y},r)}f(x)dx^1\cdots dx^n,\;\;\forall f\in C^\infty_{\text{cpt}}(\mathbb{R}^n),$$

that converges as $r\to 0$ to the current $[\![\boldsymbol{y}]\!]$ defined by the Dirac measure concentrated at $\boldsymbol{y}$.

2.2. **A baby case.** Suppose that the map F is smooth, and $M\subset \mathcal{U}$ is a smooth compact, orientable manifold with boundary of dimension $m\geq n$. Suppose that $\boldsymbol{y}\in\mathbb{R}^n$ is a regular value for $F|_M$ and $F|_{\partial M}$. Then the fiber $F^{-1}(\boldsymbol{y})\cap M$ is compact manifold of dimension $m-n$ with boundary

$$\partial\big(F^{-1}(\boldsymbol{y})\cap M\big)=F^{-1}(\boldsymbol{y})\cap\partial M.$$

This manifold is also orientable because its normal bundle in M is trivial.

For $r>0$ sufficiently small any point $\boldsymbol{z}\in B(\boldsymbol{y},r)$ is a regular value of $F|_M$ and thus

$$\mathcal{T}_{\boldsymbol{y},r}:=F^{-1}\big(\,B(\boldsymbol{y},r)\,\big)\cap M$$

is a tubular neighborhood of $F^{-1}(\boldsymbol{y})\cap M$ fibered in manifolds with boundary.

Fix an orientation $[\boldsymbol{or}_M]$ on M and denote by $F^*(\delta_{\boldsymbol{y},r})\cap[M,\boldsymbol{or}_M]$ the $(m-n)$-dimensional current in $\mathcal{U}$ given by

$$\langle\varphi,F^*(\delta_{\boldsymbol{y},r})\cap[M,\boldsymbol{or}_M]\rangle=\int_{(M,\boldsymbol{or}_M)}F^*(\Omega_{\boldsymbol{y},r})\wedge\varphi$$

$$:=\frac{1}{\boldsymbol{\omega}_n r^n}\int_{(\mathcal{T}_{\boldsymbol{y},\varepsilon},\boldsymbol{or}_M)}F^*(\Omega)\wedge\varphi,\;\;\forall\varphi\in\Omega^{m-n}_{\rm cpt}(\mathcal{U}).$$

Proposition 2.1. *If $\boldsymbol{or}_F$ is the orientation on $F^{-1}(\boldsymbol{y})\cap M$ such that $F^*\Omega\wedge\boldsymbol{or}_F=\boldsymbol{or}_M$ along $F^{-1}(\boldsymbol{y})$, then*

$$\lim_{r\to 0}F^*(\delta_{\boldsymbol{y},r})\cap[M,\boldsymbol{or}_M]=[F^{-1}(\boldsymbol{y})\cap M,\boldsymbol{or}_F]$$

weakly, i.e.,

$$\lim_{r\searrow 0}\frac{1}{\boldsymbol{\omega}_n r^n}\int_{(\mathcal{T}_{\boldsymbol{y},r},\boldsymbol{or}_M)}F^*(\Omega)\wedge\varphi=\int_{(F^{-1}(\boldsymbol{y})\cap M,\boldsymbol{or}_F)}\varphi,\;\;\forall\varphi\in\Omega^{m-n}_{\rm cpt}(\mathcal{U}).\tag{2.1}$$

Proof. For simplicity we assume that $\boldsymbol{y}$ is the origin in $\mathbb{R}^n$. We discuss first the case $m>n$. We denote by u^i the pullback of x^i to M via F. Then $F^{-1}(0)$ is described by the equalities

$$u^1=\dots=u^n=0.$$

Then

$$F^*\Omega=du^1\wedge\cdots\wedge du^n.$$

Observe that the equality (2.1) hold for all form φ such that $\operatorname{supp}\varphi\cap F^{-1}(0)=\emptyset$ because, for $r>0$ sufficiently small, the restriction of φ to the tube $\mathcal{T}_{0,r}$ is trivial. Thus we assume that $\operatorname{supp}\varphi\cap F^{-1}(0)\neq\emptyset$. Via partitions of unity we can reduce the problem to the situation when φ is supported on a tiny neighborhood $\mathcal{N}$ of a point $\boldsymbol{p}\in F^{-1}(0)$ where there exist smooth function $u^{n+1},\dots,u^m$ with the following properties,

- The collection of function $(u^1,\dots,u^n,u^{n+1},\dots,u^m)$ defines local coordinates for M on $\mathcal{N}$.
- The restrictions of $u^{n+1},\dots,u^m$ to $F^{-1}(0)\cap\mathcal{N}$ define local coordinates for $F^{-1}(0)$ on $\mathcal{N}$).
- The orientation of M is given by the m-form $du^1\wedge\cdots\wedge du^m$ and the orientation $\boldsymbol{or}_F$ is given by the $(m-n)$-form $du^{n+1}\wedge\cdots\wedge du^m$.

We write

$$\varphi=\sum_{|I|=m-n}\varphi_I du^I,\;\;\varphi_I\in C^\infty_{\rm cpt}(\mathcal{N}),\;\;I=\{1\leq i_1<\cdots<i_{m-n}\leq m\}.$$

We have

$$F^*(\Omega)\wedge\varphi=\varphi_{n+1,\dots,m}du^1\wedge\cdots\wedge du^m$$

We regard $\varphi_{n+1,\dots,m}$ as a smooth, compactly supported function on $\mathbb{R}^m$. For every $\boldsymbol{x}=(x^1,\dots,x^n)$ in $\mathbb{R}^n$ we set

$$[\varphi]_{\boldsymbol{x}} := \int_{\mathbb{R}^{n-m}} \varphi_{n+1,\dots,m}(x^1,\dots,x^n,u^{n+1},\dots,u^m)du^{m+1}\cdots du^m.$$

We deduce from the Fubini theorem that

$$\frac{1}{\boldsymbol{\omega}_n r^n}\int_{(\mathcal{T}_{\boldsymbol{y},r},\boldsymbol{or}_M)} F^*(\Omega)\wedge\varphi = \frac{1}{\boldsymbol{\omega}_n r^n}\int_{B(0,\varepsilon)}[\varphi]_{\boldsymbol{x}}\,dx^1\cdots dx^n \stackrel{r\to 0}{\longrightarrow} [\varphi]_0 = \int_{(F^{-1}(0),\boldsymbol{or}_F)}\varphi.$$

The case $m=n$ is simpler. In this case we have

$$[F^{-1}(0)\cap M,\boldsymbol{or}_F] = \sum_{\boldsymbol{p}\in F^{-1}(0)} \epsilon_{\boldsymbol{p}}\delta_{\boldsymbol{p}}$$

where $\epsilon_{\boldsymbol{p}}=\pm 1$ if $DF: T_{\boldsymbol{p}}M\to\mathbb{R}^n$ preserves/reverses orientations, and $\delta_{\boldsymbol{p}}$ denotes the Dirac measure concentrated at $\boldsymbol{p}$. □

2.3. **Lipschitzian pullbacks.** Suppose that $T\in \boldsymbol{F}_{m,K}(\mathcal{U})$, where K is a compact subset of $\mathcal{U}$. We would like to give a meaning to the current $F^*\Omega_{\boldsymbol{y},r}\cap T$ when F is only a locally Lipschitz map. To achieve this, we begin by giving a new description of this current when F is smooth. Suppose that $\eta\in\Omega^n(\mathbb{R}^n)$ is a smooth n-form. In this case, for any $\varphi\in\Omega^{m-n}_{\mathrm{cpt}}(\mathcal{U})$ we have

$$\begin{aligned}\langle\varphi, F^*\eta\cap T\rangle &= \langle F^*\eta\cup\varphi, T\rangle = (-1)^{n(m-n)}\langle\varphi\cup F^*\eta, T\rangle\\ &= (-1)^{n(m-n)}\langle F^*\eta,\varphi\cap T\rangle = (-1)^{n(m-n)}\langle\eta, F_*(\varphi\cap T)\rangle.\end{aligned}$$

Lemma 2.2. *For any smooth form $\varphi\in\Omega^{n-m}(\mathcal{U})$ the current $\varphi\cap T$ is flat and n-dimensional.*

Proof. According to Example 1.12(b) we can find compactly supported, integrable forms

$$\tau\in L^1(\mathcal{U},\Lambda^{N-m}\boldsymbol{U}^*),\;\;\sigma\in L^1(\mathcal{U},\Lambda^{N-m-1}\boldsymbol{U}^*)$$

such that

$$T=\tau\cap[\boldsymbol{U},\boldsymbol{or}_{\boldsymbol{U}}]+\partial(\sigma\cap[\boldsymbol{U},\boldsymbol{or}_U]),\;\;\boldsymbol{F}_K(T)\le\|\tau\|_{L^1(K)}+\|\sigma\|_{L^1(K)}.$$

We have

$$\varphi\cap T=(\tau\cup\varphi)\cap T+\varphi\cap\partial(\sigma\cap[\boldsymbol{U},\boldsymbol{or}_U]).$$

Note that for any $\alpha\in\Omega^n(\mathcal{U})$ we have

$$\begin{aligned}\langle\alpha,\varphi\cap\partial(\sigma\cap[\boldsymbol{U},\boldsymbol{or}_U])\rangle &= \langle\sigma\cup d(\varphi\cup\alpha),[\boldsymbol{U},\boldsymbol{or}_U]\rangle\\ &= \langle\sigma\cup d\varphi\cup\alpha),[\boldsymbol{U},\boldsymbol{or}_{\boldsymbol{U}}]\rangle\pm\langle\sigma\cup\varphi\cup d\alpha),[\boldsymbol{U},\boldsymbol{or}_{\boldsymbol{U}}]\rangle\\ &= \langle\alpha,(\sigma\cup d\varphi)\cap[\boldsymbol{U},\boldsymbol{or}_{\boldsymbol{U}}]\rangle\pm\big\langle\,\alpha,\,\partial\big(\,(\sigma\cup\varphi)\cap[\boldsymbol{U},\boldsymbol{or}_{\boldsymbol{U}}]\big)\,\big\rangle\end{aligned}$$

We deduce

$$\varphi\cap T=(\tau\cup\varphi+\sigma\cup d\varphi)\cap[\boldsymbol{U},\boldsymbol{or}_{\boldsymbol{U}}]\pm\partial\big(\,(\sigma\cup\varphi)\cap[\boldsymbol{U},\boldsymbol{or}_{\boldsymbol{U}}]\big).$$

□

The above proof implies that

$$\boldsymbol{F}_K(\varphi\cap T)\le\|\varphi\|_{L^\infty(K)}\big(\|\tau\|_{L^1(K)}+\|\sigma\|_{L^1}\,\big)+\|d\varphi\|_{L^\infty(K)}\|\tau\|_{L^1(K)},$$

for any compactly supported integrable forms

$$\tau\in L^1(\mathcal{U},\Lambda^{N-m}\boldsymbol{U}^*),\;\;\sigma\in L^1(\mathcal{U},\Lambda^{N-m-1}\boldsymbol{U}^*),$$

such that

$$T=\tau\cap[\boldsymbol{U},\boldsymbol{or}_{\boldsymbol{U}}]+\partial(\sigma\cap[\boldsymbol{U},\boldsymbol{or}_U]).$$

Invoking Proposition 1.11 we deduce from the above that

$$\boldsymbol{F}_K(\varphi\cap T)\leq \big(\|\varphi\|_{L^\infty(K)}+\|d\varphi\|_{L^\infty(K)}\big)\boldsymbol{F}_K(T)\leq 2\boldsymbol{F}_K(\varphi)\boldsymbol{F}_K(T). \tag{2.2}$$

Using the above lemma we deduce that the current $F_*(\varphi\cap T)\in\Omega_n(\mathbb{R}^n)$ is flat. Invoking Corollary 1.13 we deduce that there exists a compactly supported, integrable n-field $\xi_{T,\varphi}:\mathbb{R}^n\to\Lambda^n\mathbb{R}^n$ such that

$$F_*(\varphi\cap T)=\xi_{T,\varphi}\wedge d\mathcal{H}^n(x).$$

We can thus write for any smooth n-form $\eta\in\Omega^n(\mathbb{R}^n)$

$$\langle\varphi,F^*\eta\cap T\rangle=(-1)^{n(m-n)}\int_{\mathbb{R}^n}\eta(\xi_{T,\varphi})d\mathcal{H}^n,\ \ \forall\varphi\in\Omega^{m-n}_{\mathrm{cpt}}(\mathcal{U}). \tag{2.3}$$

Note that the right-hand-side of the above equality makes sense for locally Lipschitz maps F and bounded measurable forms η. We take (2.3) as definition of $F^*\eta\cap T$,where $\xi_{T,\varphi}$ is determined uniquely by the equality

$$\xi_{T,\varphi}\wedge d\mathcal{H}^n=F_*(\varphi\cap T).$$

We can rewrite (2.3) as

$$\big\langle\varphi,F^*(\eta\cap T)\,\big\rangle:=(-1)^{n(m-n)}\big\langle\eta,F_*(\varphi\cap T)\,\big\rangle,\ \ \forall\eta\in L^\infty\big(\mathbb{R}^n,\ \Lambda^n(\mathbb{R}^n)^*\,\big), \tag{2.4}$$

$\forall\varphi\in\Omega^{m-n}_{\mathrm{cpt}}(\mathcal{U})$.

Using (1.16) we deduce that since $F_*(\varphi\cap T)$ is flat and top dimensional it has finite mass and

$$\|F_*(\varphi\cap T)\|=\boldsymbol{F}_K(\varphi\cap T)\leq 2\max(L_K,1)L_K^n\boldsymbol{F}_K(\varphi)\boldsymbol{F}_K(T), \tag{2.5}$$

where L_F denotes the Lipschitz constant of $F|_K$. Using this in (2.4) we deduce that

$$\big\langle\,\varphi,\ F^*\eta\cap T\,\big\rangle\leq\|\eta\|_{L^\infty(\mathbb{R}^n)}\|F_*(\varphi\cap T)\|\leq 2\max(L_F,1)L_F^n\boldsymbol{F}_K(T)\|\eta\|_{L^\infty(\mathbb{R}^n)}\boldsymbol{F}_K(\varphi),$$

Hence

$$\boldsymbol{F}_K(F^*\eta\cap T)\leq 2\max(L_F,1)L_F^n\boldsymbol{F}_K(T)\|\eta\|_{L^\infty(\mathbb{R}^n)}$$

Observe next that for any $\varphi\in\Omega^{m-n-1}(\mathcal{U})$ we have

$$\big\langle\varphi,\partial\big(\,F^*\eta\cap T\big)\,\big\rangle=\langle d\varphi,F^*\eta\cap T\rangle=(-1)^{n(m-n)}\langle\eta,F_*(d\varphi\cap T)\rangle.$$

Using the identity (1.1) we deduce that

$$d\varphi\cap T=\pm\partial(\varphi\cap T)\pm\varphi\cap\partial T$$

so that

$$F_*(d\varphi\cap T)=\pm\partial F_*(\varphi\cap T)\pm F_*(\varphi\cap\partial T).$$

Note that $F_*(\varphi\cap T)\in\Omega_{n+1}(\mathbb{R}^n)=\{0\}$ so that

$$F_*(d\varphi\cap T)=\pm F_*(\varphi\cap\partial T), \tag{2.6}$$

which shows that

$$\partial\big(\,F^*\eta\cap T\,\big)=\pm\eta\cap\partial T. \tag{2.7}$$

The correct sign is determined from (1.1) by assuming that η is a smooth n-form on $\mathbb{R}^n$ and observing that $dF^*\eta=F^*d\eta=0$.

Note that if T is normal, then so is $\varphi\cap T$ and

$$\|\varphi\cap T\|+\|\partial(\varphi\cap T)\|\leq\|\varphi\|_{L^\infty(K)}\big(\|T\|+\|\partial T\|\big).$$

We deduce that $F^*\eta\cap T$ is also normal and

$$\|F^*\eta\cap T\|_K+\|\partial(F^*\eta\cap T)\|_K\leq\|\eta\|_{L^\infty(F(K))}\big(\,\|T\|+\|\partial T\|\,\big). \tag{2.8}$$

Using (1.14) and (2.6) we deduce that if T is normal, $G:\mathcal{U}\to\mathbb{R}^n$ is another locally Lipschitz constant and $L_{F,G}$ is the largest of the Lipschitz constants of $F|_K$ and $G|_K$, then for any $\eta\in\Omega^{m-n}(\mathcal{U})$

$$\|F_*(\varphi\cap T)-G_*(\varphi\cap T)\| = \boldsymbol{F}\big(F_*(\varphi\cap T)-G_*(\eta\cap T)\big)$$

$$\leq \|F-G\|_{C^0(K)}\big(L^n_{F,G}\|\varphi\cap T\|+L^{n-1}_{F,G}\|\varphi\cap\partial T\|\big)$$

$$\leq \|F-G\|_{C^0(K)}\|\varphi\|_{L^\infty(K)}\big(L^n_{F,G}\|T\|+L^{n-1}_{F,G}\|\partial T\|\big).$$

We deduce as before

$$\boldsymbol{F}_K\big(F^*\eta\cap T-G^*\eta\cap T\big)\leq \|F-G\|_{C^0(K)}\|\eta\|_{L^\infty(K)}\big(L^n_{F,G}\|T\|+L^{n-1}_{F,G}\|\partial T\|\big), \tag{2.9}$$

$\forall T\in\boldsymbol{N}_{m,K}(\mathcal{U})$. We gather all of the above observations in our next result.

Proposition 2.3. *(a) If $T\in\boldsymbol{F}_{m,K}(\mathcal{U})$ and η is a bounded, measurable n-form on $\mathbb{R}^n$, then $F^*\cap\eta\in\boldsymbol{F}_{m-n,K}(\mathcal{U})$. If additionally $T\in\boldsymbol{N}_{m,K}(\mathcal{U})$, then $F^*\cap\eta\in\boldsymbol{N}_{m-n,K}(\mathcal{U})$*

(b) If $T\in\boldsymbol{F}_{M,K}(\mathcal{U})$, $m\geq n$ and η_j are bounded measurable n-forms on $\mathbb{R}^n$ such that

$$\lim_{j\to\infty}\eta_j(y)=\eta(y)\ \ \textit{a.e. } y\in F(K),$$

$$\sup_j\|\eta_j\|_{L^\infty(K)}<\infty,$$

then

$$\lim_{j\to\infty}\boldsymbol{F}_K\big(F^*\eta_j\cap T-F^*\eta\cap T\big)=0.$$

(c) If

$$\lim_{j\to\infty}\boldsymbol{F}_K(T_j-T)=0,$$

and η is a bounded, measurable n-form on $\mathbb{R}^n$ then

$$\lim_{j\to\infty}\boldsymbol{F}_K\big(F^*\eta\cap T_j-F^*\eta\cap T\big)=0.$$

(d) If $T\in\boldsymbol{N}_{m,K}(\mathcal{U})$, η is a bounded, measurable n-form on $\mathbb{R}^n$ and $F_j:\mathcal{U}\to\mathbb{R}^n$ are smooth maps such that

$$\lim_{j\to\infty}\|F_j-F\|_{C^0(K)}=0\ \ \textit{and}\ \ \sup_j\|DF_j\|_{C^0(K)}<\infty,$$

then

$$\lim_{j\to\infty}\boldsymbol{F}_K\big(F_j^*\eta\cap T-F^*\eta\cap T\big)=0. \qquad\square$$

2.4. **Slicing.** Suppose that $T\in\boldsymbol{F}_{m,K}(\mathcal{U})$, $m\geq n$. For any $\boldsymbol{y}\in\mathbb{R}^n$ and any $r>0$ we set

$$F^{-1}(\boldsymbol{y})\cap_r T:=F^*\Omega_{\boldsymbol{y},r}\cap T.$$

We will sometime use the alternate notation

$$\langle T,F,\boldsymbol{y}\rangle_r:=F^{-1}(\boldsymbol{y})\cap_r T.$$

We obtain in this fashion a continuous map

$$\mathbb{R}^n\ni\boldsymbol{y}\mapsto F^{-1}(\boldsymbol{y})\cap_\varepsilon T\in\boldsymbol{F}_{m-n,K}(\mathcal{U}).$$

Theorem 2.4. *Let $T \in \boldsymbol{F}_{m,K}\mathcal{U})$ and $F : \mathcal{U} \to \mathbb{R}^n$ a locally Lipschitzian map, $n \leq m$.*
(a) There exists a negligible set $Z \subset \mathbb{R}^n$ such that for any $\boldsymbol{y} \in \mathbb{R}^n \setminus Z$ the weak limit

$$\lim_{r \searrow 0} F^{-1}(\boldsymbol{y}) \cap_r T$$

exists. It defines a current denoted by $F^{-1}(\boldsymbol{y}) \cap T \in \Omega_{n-m}(\mathcal{U})$ supported on $F^{-1}(\boldsymbol{y}) \cap K$ and called the F-slice of T over $\boldsymbol{y}$. *We will some time use the alternate notation $\langle T, F, \boldsymbol{y}\rangle$ to denote the F slice over $\boldsymbol{y}$.*
(b) For $\varphi \in \Omega^{m-n}(\mathcal{U})$ we have

$$\big\langle\, \eta, F^{-1}(\boldsymbol{z}) \cap T \,\big\rangle = \Omega(\xi_{T,\varphi}(\boldsymbol{z}))\ \ \textit{a.e. } \boldsymbol{z} \in \mathbb{R}^n,$$

where $\Omega = dx^1 \wedge \cdots \wedge dx^n \in \Lambda^n(\mathbb{R}^n)^*$.

Proof. For every $\varphi \in \Omega^{m-n}_{\mathrm{cpt}}(\mathcal{U})$ we have

$$\langle \varphi, F^{-1}(\boldsymbol{y}) \cap_r T\rangle = \frac{1}{\boldsymbol{\omega}_n r^n}\Big\langle\, \mathbb{1}_{B(\boldsymbol{y},r)}\Omega, F_*(\varphi \cap T)\,\Big\rangle = \frac{1}{\boldsymbol{\omega}_n r^n}\int_{B(\boldsymbol{y},r)} \Omega(\xi_{T,\varphi}) d\mathcal{H}^n.$$

We need to extract some additional information about $\xi_{T,\varphi}$.

Using Proposition 1.15 we can find currents $R \in \boldsymbol{F}_{m,U}(\mathcal{O})$, $S \in \boldsymbol{F}_{m+1,K}(\mathcal{U})$ of finite mass such that

$$T = R + \partial S,\ \ \boldsymbol{F}_K(T) = \|R\| + \|S\|.$$

For $\varphi \in \Omega^{m-n}_{\mathrm{cpt}}(\mathcal{U})$ we have

$$\varphi \cap T = \varphi \cap R + \varphi \cap \partial S.$$

If F is smooth, then for any $\omega \in \Omega^n_{\mathrm{cpt}}(\mathbb{R}^n)$ we have

$$\langle \omega, F_*(\varphi \cap T)\rangle = \langle F^*\omega, \varphi \cap T\rangle = \langle F^*\omega, \varphi \cap R\rangle + \langle F^*\omega, \varphi \cap \partial S\rangle$$

($d\omega = 0$)

$$= \langle \varphi \cup F^*\omega, R\rangle + \langle d\varphi \cup F^*\omega, S\rangle = \int_{\mathcal{U}} \langle \varphi \cup F^*\omega, \vec{R}\rangle d\mu_R + \int_{\mathcal{U}} \langle d\varphi \cup F^*\omega, \vec{S}\rangle d\mu_S.$$

For every $\boldsymbol{y} \in \mathbb{R}^n$ we set

$$\rho_R(y) = \sup_{F(\boldsymbol{p})=\boldsymbol{y}} \big|\langle\, (\varphi \cup F^*\omega)_{\boldsymbol{p}}, \vec{R}_{\boldsymbol{p}}\,\rangle\big|,\ \ \rho_S(y) = \sup_{F(\boldsymbol{p})=\boldsymbol{y}} \big|\langle\, (d\varphi \cup F^*\omega)_{\boldsymbol{p}}, \vec{S}_{\boldsymbol{p}}\,\rangle\big|.$$

We deduce

$$\begin{aligned}\big|\langle \omega, F_*(\varphi \cap T)\rangle\big| &\leq \int_{\mathcal{U}} F^*\rho_R d\mu_R + \int_{\mathcal{U}} F^*\rho_S d\mu_S \\ &= \int_{\mathbb{R}^n} \rho_R(\boldsymbol{y}) dF_*\mu_R(\boldsymbol{y}) + \int_{\mathbb{R}^n} \rho_S(\boldsymbol{y}) dF_*\mu_S(\boldsymbol{y}).\end{aligned}$$

There exists a universal constant $C_0 > 0$ such that

$$\rho_R(\boldsymbol{y}) \leq CL_F^m \|\varphi\|_{L^\infty} |\omega_{\boldsymbol{y}}|,\ \ \rho_R(\boldsymbol{y}) \leq CL_F^{m+1} \|d\varphi\|_{L^\infty} |\omega_{\boldsymbol{y}}|,\ \ \forall \varphi,\ \ \forall \omega,$$

where L_F denotes the Lipschitz constant of $F|_K$. We deduce

$$\big|\, \langle \omega, F_*(\varphi \cap T)\rangle\,\big| \leq C_0 \|\varphi\|_{C^1(K)}\| \left(L_F^m \int_{\mathbb{R}^n} |\omega_{\boldsymbol{y}}| dF_*\mu_R(\boldsymbol{y}) + L_F^{m+1}\int_{\mathbb{R}^n} |\omega_{\boldsymbol{y}}| dF_*\mu_S(\boldsymbol{y}) \right). \tag{2.10}$$

An approximation argument shows that the above inequality continues to hold for any locally Lipschitz map F and any bounded measurable n form ω.

Using [6, Thm. 4.7] we deduce that there exists a negligible subset $\Delta \subset \mathbb{R}^n$ such that for any $\boldsymbol{y} \in \mathbb{R}^n \setminus \Delta$ the limits

$$\Theta_R(\boldsymbol{y}) := \lim_{r\searrow 0} \frac{1}{\boldsymbol{\omega}_n r^n} F_*\mu_R\big(\, B(\boldsymbol{y}, r)\,\big),\;\; \Theta_S(\boldsymbol{y}) := \lim_{r\searrow 0} \frac{1}{\boldsymbol{\omega}_n r^n} F_*\mu_S\big(\, B(\boldsymbol{y}, r)\,\big)$$

exits and are finite. For $\varepsilon > 0$ we set

$$\Theta_T(\boldsymbol{y}, r) := \frac{1}{\boldsymbol{\omega}_n r^n} F_*\mu_R\big(\, B(\boldsymbol{y}, r)\,\big) + \frac{1}{\boldsymbol{\omega}_n r^n} F_*\mu_S\big(\, B(\boldsymbol{y}, r)\,\big).$$

Fix a countable subset $\mathcal{F} \subset \Omega^{m-n}_{\rm cpt}(\mathcal{U})$ such that for any $\varphi \in \Omega^{m-n}(\mathcal{U})$ there exits a sequence φ_j in $\mathcal{F}$ such that

$$\lim_{j\to\infty} \|\varphi_j - \varphi\|_{C^1(K)} = 0.$$

This implies classically (see e.g. [6, Thm. 4.7], [7, Thm 11.1]) that there exists a negligible subset $Z_\varphi \subset \mathbb{R}^n$ such that for any $\boldsymbol{y} \in \mathbb{R}^n \setminus Z_\varphi$ the limit

$$\lim_{r\searrow 0} \langle \varphi, F^{-1}(\boldsymbol{y}) \cap_r T\rangle = \lim_{r\searrow 0} \frac{1}{\boldsymbol{\omega}_n r^n} \Big\langle\, \mathbb{1}_{B(\boldsymbol{y},r)}\Omega, F_*(\varphi\cap T)\,\Big\rangle = \lim_{r\searrow 0} \frac{1}{\boldsymbol{\omega}_n r^n} \int_{B(\boldsymbol{y},r)} \Omega(\xi_{T,\varphi}) d\mathcal{H}^n$$

exists and it is equal to $\Omega(\xi_{T,\varphi}(\boldsymbol{y})\rangle$. The set

$$Z := \Delta \cup \left(\bigcup_{\varphi\in\mathcal{F}} Z_\varphi \right)$$

is negligible. We deduce from (2.10) that there exists a constant $C_1 > 0$ such that, for any $\varphi_0, \varphi_1 \in \Omega^{n-m}_{\rm cpt}(\mathcal{U})$, any $r > 0$, and any $\boldsymbol{y} \in \mathbb{R}^n \setminus Z$ we have

$$\big|\, \langle \varphi_0, F^{-1}(\boldsymbol{y}) \cap_r T\rangle - \langle \varphi_1, F^{-1}(\boldsymbol{y}) \cap_r T\rangle \,\big| \leq C\|\varphi_0 - \varphi_1\|_{C^1} \Theta_T(\boldsymbol{y}, r). \tag{2.11}$$

To finish (a) it suffices to prove that for any $\boldsymbol{y} \in \mathbb{R}^n \setminus Z$ and any $\varphi \in \Omega^{m-n}_{\rm cpt}(\mathcal{U})$ the function

$$(0, 1] \ni r \mapsto \langle A_\varphi(r) := \langle \varphi, F^{-1}(\boldsymbol{y}) \cap_r T\rangle \in \mathbb{R}$$

as a finite limit as $r \to 0$. We will achieve this by showing that for any $\varepsilon > 0$ there exists $r = r(\varepsilon) > 0$ such that

$$|A_\varphi(r') - A_\varphi(r'')| < \varepsilon,\;\; \forall 0 < r', r'' < r(\varepsilon).$$

Set

$$M_{\boldsymbol{y}} := \sup_{r\in(0,1)} \Theta_T(\boldsymbol{y}, r),$$

and choose $\varphi' \in \mathcal{F}$ such that

$$M_{\boldsymbol{y}} \|\varphi - \varphi'\|_{C^1(K)} < \frac{\varepsilon}{3}. \tag{2.12}$$

Since the limit $\lim_{r\searrow 0} A_{\varphi'}(r)$ exists and it is finite, there exists $r(\varepsilon) \in (0, 1)$ such that

$$|A_{\varphi'}(r') - A_{\varphi'}(r'')| < \frac{\varepsilon}{3},\;\; \forall 0 < r', r'' < r(\varepsilon).$$

Then

$$|A_\varphi(r') - A_\varphi(r'')| \leq |A_\varphi(r') - A_{\varphi'}(r')| + |A_{\varphi'}(r') - A_{\varphi'}(r'')| + |A_{\varphi'}(r'') - A_\varphi(r'')|$$

(use (2.11))

$$\leq \|\varphi - \varphi'\|_{C^1(K)} \Theta_T(\boldsymbol{y}, r') + \frac{\varepsilon}{3} + \|\varphi - \varphi'\|_{C^1(K)} \Theta_T(\boldsymbol{y}, r'') \stackrel{(2.12)}{<} \varepsilon.$$

This proves (a). Part (b) follows from classical density results, e.g. [6, Thm. 4.7] or [7, Thm 11.1]. □

The ideas in the above proof lead to the following additional information about slices. For details we refert to [3, Thm. 4.3.2].

Theorem 2.5. *Under the same assumptions as in Theorem 2.4 the following hold for any function* $\Phi \in L^\infty(\mathbb{R}^n)$.
(a) For any $\varphi \in \Omega^{m-n}_{\mathrm{cpt}}(\mathcal{U})$ *we have*

$$\int_{\mathbb{R}^n} \Phi(\boldsymbol{y})\langle \varphi, F^{-1}(\boldsymbol{y}) \cap T\rangle = \langle \varphi, F^*(\Phi\Omega) \cap T\rangle.$$

(b) If $\|T\| < \infty$ *then*

$$f^*(\Phi\Omega) \cap T = F^*(\Phi) \cap (F^*\Omega \cap T),$$

$$\int_{\mathbb{R}^n} \Phi(\boldsymbol{y})\|F^{-1}(\boldsymbol{y}) \cap T\| d\mathcal{H}^n(\boldsymbol{y}) = \|f^*(\Phi\Omega) \cap T\|$$

$$\leq L_F^n \int_{\mathcal{U}} F^*(\Phi) d\mu_T,$$

$$\int_{\mathbb{R}^n} \left(\int_{\mathcal{U}} f d\mu_{F^{-1}(\boldsymbol{y})\cap T} \right) d\mathcal{H}^n(\boldsymbol{y}) = \int_{\mathcal{U}} f(\boldsymbol{u}) d\mu_{F^*\Omega\cap T}(\boldsymbol{u}), \;\; \forall f \in L^\infty(\mathcal{U}).$$

(c) If $T \in \boldsymbol{N}_{m,K}(\mathcal{U})$ *then* $F^{-1}(\boldsymbol{y}) \cap T \in \boldsymbol{N}_{m-n,K}(\mathcal{U})$ *for a.e.* $\boldsymbol{y} \in \mathbb{R}^n$.
(d) $F^{-1}(\boldsymbol{y}) \cap T \in \boldsymbol{F}_{m-n,K}(\mathcal{U})$ *for a.e.* $\boldsymbol{y} \in \mathbb{R}^n$.
(e) If K *is a Lipschitz neighborhood retract in* $\mathcal{U}$*, then the function*

$$\mathbb{R}^n \ni \boldsymbol{y} \mapsto F^{-1}(\boldsymbol{y}) \cap T \in \boldsymbol{F}_{m-n,K}(\mathcal{U}),$$

is summable with respect to the norm $\boldsymbol{F}_K$ *and*

$$\boldsymbol{F}_K\big(F^{-1}(\boldsymbol{y}) \cap T - F^{-1}(\boldsymbol{y}) \cap_r T \big) \leq \int_{B(\boldsymbol{y},r)} \boldsymbol{F}_K\big(F^{-1}(\boldsymbol{y}) \cap T - F^{-1}(\boldsymbol{z}) \cap T \big) d\mathcal{H}^n(\boldsymbol{z}) \to 0,$$

as $\rho \searrow 0$*, for a.e.* $\boldsymbol{y} \in \mathbb{R}^n$.
(f) If $G : \mathbb{R}^n \to \mathbb{R}^n$ *is locally Lipschitz, then*

$$(G \circ F)^{-1}(\boldsymbol{z}) \cap T = \sum_{\boldsymbol{y} \in G^{-1}(\boldsymbol{z})} \text{sign}\, \det D_{\boldsymbol{y}} G \cdot F^{-1}(\boldsymbol{y}) \cap T.$$

(g) If $\mathcal{V}$ *is an open subset of a finite dimensional Euclidean space* $\boldsymbol{V}$ *and* $G : \mathcal{U} \to \mathcal{V}$*, and* $H : \mathcal{V} \to \mathbb{R}^n$ *are locally Lipschitz maps, then*

$$G_*\big((H \circ G)^*(\Phi\Omega) \cap T \big) = H^*(\Phi\Omega) \cap G_*(T),$$
$$G_*\big((H \circ G)^{-1}(\boldsymbol{y}) \cap T \big) = H^{-1}(\boldsymbol{y}) \cap G_* T \;\; \textit{for a.e. } \boldsymbol{y} \in \mathbb{R}^n.$$

□

Theorem 2.6. *Suppose that* $F : \mathcal{U} \to \mathbb{R}^n$ *and* $G : \mathcal{U} \to \mathbb{R}^\nu$ *are locally Lipchitz maps and* $T \in \boldsymbol{F}_{m,K}(\mathcal{U})$*,* $m \geq n + \nu$*. Define the cartesian product*

$$F \times G : \mathcal{U} \to \mathbb{R}^n \times \mathbb{R}^\nu, \;\; (F \times G)(\boldsymbol{u}) = \big(F(\boldsymbol{u}), G(\boldsymbol{u}) \big)$$

Then for a.e. $(\boldsymbol{y}, \boldsymbol{z}) \in \mathbb{R}^n \times \mathbb{R}^\nu$ *we have*

$$(F \times G)^{-1}(\boldsymbol{y}, \boldsymbol{z}) \cap T = G^{-1}(\boldsymbol{z}) \cap \big(F^{-1}(\boldsymbol{y}) \cap T \big).$$

□

2.5. **An alternative approach to slicing.** Suppose that $f:\mathcal{U}\to\mathbb{R}$ is a locally Lipschitz map and $T\in\mathcal{N}_{m,K}(\mathcal{U})$ is a normal current of dimension $m\geq 1$. For $r\in\mathbb{R}$ we set

$$\begin{aligned}\langle T,f,r+\rangle &:= (\partial T)|_{\{f>r\}}-\partial\big(T|_{\{f>r\}}\big),\\ \langle T,f,r-\rangle &:= \partial\big(T|_{\{f<r\}}\big)-(\partial T)|_{\{f<r\}}\\ &= (\partial T)|_{\{f\geq r\}}-\partial\big(T|_{\{f\geq r\}}\big).\end{aligned}$$

For all but countably many r-s we have

$$\mu_T\big(\{f=0\}\big)+\mu_{\partial T}\big(\{f=0\}\big)=0$$

so that

$$\langle T,f,r+\rangle=\langle T,f,r-\rangle$$

for all but countable many $r\in\mathbb{R}$.

Proposition 2.7. *For almost all $r\in\mathbb{R}$ we have*

$$f^{-1}(r)\cap T=\frac{1}{2}\big(\langle T,f,r+\rangle+\langle T,f,r-\rangle\big). \tag{2.13}$$

Proof. Let us first observe that for every Lipschitz function $\gamma:\mathbb{R}\to\mathbb{R}$ we have the equality

$$f^*(d\gamma)\cap T=f^*(\gamma)\cap\partial T-\partial\big(f^*(\gamma)\cap T).$$

Indeed, this is true for smooth f and g, and by smoothing we can extend this to general f and γ. Note that $d\gamma$ is a bounded measurable form on $\mathbb{R}$ and $f^*(d\gamma)\cap T$ can be defined as in Subsection 2.3. Define $\gamma_{r,h}:\mathbb{R}\to\mathbb{R}$ to be the Lipschitz approximation of the Heaviside function $t\mapsto H(t-r)$ depicted in Figure 1

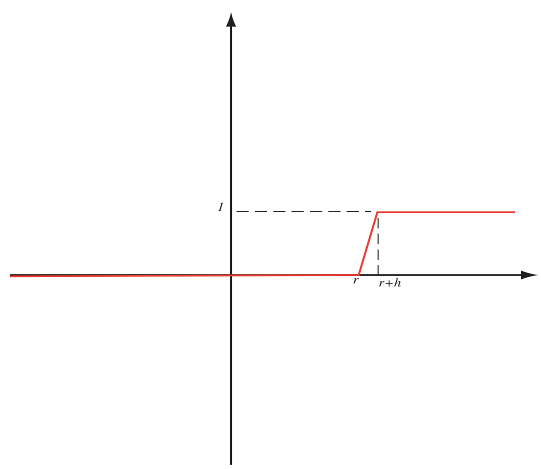

FIGURE 1. A Lipschitzian approximation of the Heaviside function

Then

$$\mathbb{1}_{[r+h,\infty)}\leq\gamma_{r,h}\leq\mathbb{1}_{[r,\infty)},\;\; d\gamma_{h,r}=\frac{1}{h}\mathbb{1}_{(r,r+h]}dt,$$

and

$$\frac{1}{h}f^*dt\cap T|_{r<f<r+h}-\langle T,f,r+\rangle$$

$$= f^*(d\gamma_{r,h})\cap T - \langle T,f,r+\rangle = \underbrace{\bigl(\gamma_{r,h}\circ f - \mathbb{1}_{\{f>r\}}\bigr)(\partial T)}_{=:R} - \partial\bigl(\underbrace{(\gamma_{r,h}\circ f - \mathbb{1}_{\{f>r\}})T}_{=:-S}\bigr)$$

Hence

$$\boldsymbol{F}\bigl(f^*(d\gamma_{r,h})\cap T - \langle T,f,r+\rangle\bigr) \leq \|R\| + \|S\| \leq \bigl(\mu_{\partial T} + \mu_T\bigr)(\{r<f<r+h\}).$$

It follows that for almost all r we have

$$\lim_{h\searrow 0}\boldsymbol{F}\Bigl(\frac{1}{h}f^*dt\cap T|_{r<f<r+h} - \langle T,f,r+\rangle\Bigr) = \lim_{h\searrow 0}\bigl(\mu_{\partial T}+\mu_T\bigr)(\{r<f<r+h\}) = 0$$

We obtain similarly that, for almost all r, we have

$$\lim_{h\searrow 0}\boldsymbol{F}\Bigl(\frac{1}{h}f^*dt\cap T|_{r-h<f<r} - \langle T,f,r-\rangle\Bigr) = 0.$$

To conclude observe that for almost all r we have

$$\frac{1}{2h}\bigl(f^*dt\cap T|_{r-h<f<r} + f^*dt\cap T|_{r<f<r+h}\bigr) = f^{-1}(r)\cap_h T.$$

□

Combining the above proposition with Theorem 2.6 we can produce alternative description of the type (2.13) for normal currents $T\in\boldsymbol{N}_m(\mathcal{U})$, and locally Lipschitz maps $f:\mathcal{U}\to\mathbb{R}^n$, $1\leq n\leq m$.

For integer multiplicity rectifiable currents we can give an even more explicit description of the slicing process. Recall that a current $T\in\Omega_m(\mathcal{U})$ is called rectifiable if the following hold.

- It is representable by integration.
- There exists a countably m-rectifiable set $M = M_T\subset\mathcal{U}$ and a locally $\mathcal{H}^m$-integrable function $\theta_T: M\to\mathbb{R}$ such that
$$\mu_T = \theta_T\cdot\mathcal{H}^m|_M.$$
- For $\mathcal{H}^m$-a.e. point $\boldsymbol{p}\in M$ there exists a basis $\boldsymbol{e}_1(\boldsymbol{p}),\dots,\boldsymbol{e}_m(\boldsymbol{p})$ of the approximate tangent space $T_{\boldsymbol{p}}M$ such that
$$\vec{T} = \boldsymbol{e}_1(\boldsymbol{p})\wedge\cdots\wedge\boldsymbol{e}_m(\boldsymbol{p}).$$

We denote this current by $[\![M,\vec{T},\theta]\!]$. The rectifiable current T is said to have *integer multiplicity* if $\theta_T(\boldsymbol{p})\in\mathbb{Z}$ for $\mathcal{H}^m$ -a.e. $\boldsymbol{p}\in M$.

Proposition 2.8. *Suppose that* $T = [\![M,\vec{T},\theta]\!]\in\Omega_m(\mathcal{U})$ *is a* normal *integer multiplicity rectifiable* m*-curent. We set* $f_M := f|_M$,

$$M_* := \bigl\{\boldsymbol{p}\in M;\ \nabla f|_M(\boldsymbol{p})\neq 0\bigr\},$$

(a) For almost all $r\in\mathbb{R}$ *the set*

$$M_r := f^{-1}(r)\cap M^*$$

is countably $(m-1)$*-rectifiable, and for* $\mathcal{H}^{m-1}$*-a.e.* $\boldsymbol{p}\in M_r$ *both* $T_{\boldsymbol{p}}M_r$ *and* $\nabla f_M(\boldsymbol{p})$ *exist. Moreover, for such* r *and* $\boldsymbol{p}$ *the approximate tangent space* $T_{\boldsymbol{p}}M_r$ *exists and*

$$T_{\boldsymbol{p}}M = T_{\boldsymbol{p}}M_r\oplus\mathbb{R}\langle\nabla f_M\rangle.$$

We define $\vec{T}_r(\boldsymbol{p})\in\Lambda^{m-1}\boldsymbol{U}$ *by the equality*

$$(\vec{T}_r(\boldsymbol{p}),\xi) = \left(\vec{T}_{\boldsymbol{p}},\frac{1}{|\nabla f_M(\boldsymbol{p})|}\nabla f_M(\boldsymbol{p})\wedge\xi\right),\quad\forall\xi\in\Lambda^{m-1}(\boldsymbol{U}),$$

and we set

$$\theta_r(\boldsymbol{p}) = \begin{cases} \theta(\boldsymbol{p}), & f(\boldsymbol{p}) = r, \;\; \nabla f_M(\boldsymbol{p}) \neq 0 \\ 0, & \textit{otherwise}. \end{cases}$$

(b) For almost any $r \in \mathbb{R}$ we have

$$f^{-1}(r) \cap T = [\![M_r, \vec{T}_r, \theta_r]\!].$$

Proof. Part (a) follows from the basic properties of countably rectifiable sets. We refer to [6, Lemma 28.1] for more details.

To prove (b) we fix a countable subset $\mathcal{F} \subset \Omega^{m-1}_{\mathrm{cpt}}(\mathcal{U})$ that is dense in the $C^1(\operatorname{supp} T)$-norm and we prove that there exists a negligible subset $Z \subset \mathbb{R}$ wwith the following properties.

(i) For any $r \in \mathbb{R} \setminus Z$ both $f^{-1}(r) \cap T$ and $[\![M_r, \vec{T}_r, \theta|_{M_r}]\!]$ are well defined.
(ii) For any $\varphi \in \mathcal{F}$ and any $r \in \mathbb{R} \setminus Z$ we have

$$\langle \varphi, f^{-1}(r) \cap T \rangle = \langle \varphi, [\![M_r, \vec{T}_r, \theta|_{M_r}]\!] \rangle. \tag{2.14}$$

Fix a negligible subset $Z_0 \subset \mathbb{R}$ such that $r \in \mathbb{R} \setminus Z_0$ both $f^{-1}(r) \cap T$ and $[\![M_r, \vec{T}_r, \theta|_{M_r}]\!]$ are well defined. We will show that for any $\varphi \in \mathcal{F}$ there exists a negligible subset $Z_\varphi \subset \mathbb{R} \setminus Z_0$ such that (refeq: slice-rect) holds for any $r \in \mathbb{R} \setminus (Z_0 \cup Z_\varphi)$. More precisely we have to show that

$$\langle \varphi, [\![M_r, \vec{T}_r, \theta|_{M_r}]\!] \rangle = \lim_{h \searrow 0} \frac{1}{h} \langle \mathbb{1}_{\{r \leq f \leq r+h\}} f^* dt \wedge \varphi, T \rangle. \tag{2.15}$$

We have

$$\Big\langle \, \mathbb{1}_{\{r \leq f \leq r+h\}} f^* dt \wedge \varphi, T \, \Big\rangle = \Big\langle \, \mathbb{1}_{\{r \leq f \leq r+h\} \cap M} df \wedge \varphi, T \, \Big\rangle$$

$$= \int_{\{r \leq f \leq r+h\} \cap M} |\nabla f_M(\boldsymbol{p})| \underbrace{\left(\frac{1}{|\nabla f_M(\boldsymbol{p})|} df_M \wedge \varphi \right)_{\boldsymbol{p}} (\vec{T}_{\boldsymbol{p}}) \theta(\boldsymbol{p})}_{= g_\varphi(\boldsymbol{p})} \, d\mathcal{H}^m(\boldsymbol{p})$$

(use the co-area formula)

$$= \int_r^{r+h} \left(\int_{M_t} g_\varphi(\boldsymbol{p}) d\mathcal{H}^{m-1}(\boldsymbol{p}) \right) .dt$$

Now observe that

$$\int_{M_t} g_\varphi(\boldsymbol{p}) d\mathcal{H}^{m-1}(\boldsymbol{p}) = \int_{M_t} \varphi_{\boldsymbol{p}}(\vec{T}_t(\boldsymbol{p})) \theta(\boldsymbol{p}) d\mathcal{H}^{m-1}(\boldsymbol{p}) = \langle \varphi, [\![M_r, \vec{T}_t, \theta_t]\!] \rangle$$

Hence

$$\frac{1}{h} \langle \mathbb{1}_{\{r \leq f \leq r+h\}} f^* dt \wedge \varphi, T \rangle = \frac{1}{h} \int_r^{r+h} \langle \varphi, [\![M_r, \vec{T}_t, \theta_t]\!] \rangle dt$$

To prove (2.15) for a.e. r it thus suffices to show that the function

$$t \mapsto \langle \varphi, [\![M_r, \vec{T}_t, \theta_t]\!] \rangle$$

is locally integrable. This is another application of the co-area formula

$$\int_{\mathbb{R}} \| [\![M_r, \vec{T}_t, \theta_t]\!] dt = \int_{\mathbb{R}} \left(\int_{M_r} |\theta_r| d\mathcal{H}^{m-1} \right) = \int_M |\nabla f_M| |\theta| d\mathcal{H}^m$$

$$\leq \| \nabla f_M \|_{L^\infty} \int_M |\theta| d\mathcal{H}^m = \| \nabla f_M \|_{L^\infty} \| T \|.$$

□

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