THE BLOWUP ALONG THE DIAGONAL OF THE SPECTRAL FUNCTION OF THE LAPLACIAN

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ABSTRACT. We formulate a precise conjecture about the universal behavior near the diagonal of the spectral function of the Laplacian of a smooth compact Riemann manifold. We prove this conjecture when the manifold and the metric are real analytic, and we also present an alternate proof when the manifold is the round sphere.

CONTENTS

1. INTRODUCTION

Suppose that (M, g) is a compact, connected m-dimensional Riemannian manifold, and $(\Psi_n)_{n\geq 0}$ is a (complete) orthonormal basis of $L^2(M, g)$ consisting of eigenfunctions of Δ_g

$$\Delta_g \Psi_n = \lambda_n \Psi_n,\;\; 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots .$$

For every $L > 0$ we define the spectral function $\mathcal{E}_L : M \times M \to \mathbb{R}$ by

$$\mathcal{E}_L(\boldsymbol{p}, \boldsymbol{q}) = \sum_{\lambda_n \leq L} \Psi_n(\boldsymbol{p})\Psi_n(\boldsymbol{q}).$$

Equivalently, $\mathcal{E}_L$ is the Schwartz kernel of the orthogonal projection onto

$$H_L := \bigoplus_{\lambda \leq L} \ker(\lambda - \Delta).$$

This shows that as $L \to \infty$ the spectral function $\mathcal{E}_L$ converges in the sense of distributions to the Dirac-type distribution supported by the diagonal

$$\mathcal{D}_M = \big\{ (\boldsymbol{p}, \boldsymbol{q}) \in M \times M;\;\; \boldsymbol{p} = \boldsymbol{q} \,\big\}.$$

The goal of this paper is to describe a universal law governing the behavior of $\mathcal{E}_L$ as $L \to \infty$ in an infinitesimal neighborhood of the diagonal. Here are the specifics.

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We denote by $\mathcal{N}$ the normal bundle of the diagonal embedding. For any $\boldsymbol{p}\in M$ we denote by $\mathcal{N}_{\boldsymbol{p}}$ the fiber over $(\boldsymbol{p},\boldsymbol{p})\in\mathcal{D}_M$. Also we let $r:\mathcal{N}\to\mathbb{R}$ denote the radial distance function along the fibers of $\mathcal{N}$, and we set

$$\mathcal{N}^R:=r^{-1}\big([0,R)\big),\;\; \mathcal{N}^R_{\boldsymbol{p}}:=\mathcal{N}^R\cap\mathcal{N}_{\boldsymbol{p}},\;\;\forall R>0,\;\;\boldsymbol{p}\in M.$$

In other words, $\mathcal{N}^R\subset\mathcal{N}$ is the associated bundle of normal disks of radius R. If $\hbar$ is sufficiently small, then the exponential map induces a diffeomorphism from $\mathcal{N}^\hbar$ onto an open neighborhood $\mathcal{U}^\hbar$ of the diagonal. Fix once and for all such a $\hbar$. We denote by $\mathcal{E}^\hbar_L$ the pullback of $\mathcal{E}_L|_{\mathcal{U}^\hbar}$ to $\mathcal{N}^\hbar$.

For every positive real number λ we denote by $\mathcal{M}_\lambda:\mathcal{N}\to\mathcal{N}$ the rescaling map described on $\mathcal{N}_{\boldsymbol{p}}$ by

$$\mathcal{N}_{\boldsymbol{p}}\ni\boldsymbol{v}\mapsto\frac{1}{\lambda}\boldsymbol{v}\in\mathcal{N}_{\boldsymbol{p}}.$$

We define

$$\bar{\mathcal{E}}_L:\mathcal{N}^{L^{1/2}\hbar}\to\mathbb{R},\;\;\bar{\mathcal{E}}_L=L^{-\frac{m}{2}}\mathcal{M}^*_{L^{\frac{1}{2}}}\mathcal{E}^\hbar_L.$$

For any $\boldsymbol{p}\in M$ we denote by $\bar{\mathcal{E}}_{L,\boldsymbol{p}}$ the restriction of $\bar{\mathcal{E}}_L$ to the fiber $\mathcal{N}_{\boldsymbol{p}}$.

The Universality Conjecture. There exists $\rho>0$ such that for any $\boldsymbol{p}\in M$ the functions

$$\bar{\mathcal{E}}_{L,\boldsymbol{p}}:\mathcal{N}_{\boldsymbol{p}}^{L^{\frac{1}{2}}\hbar}\to\mathbb{R}$$

converge as $L\to\infty$ in the topology of $C^\infty(\mathcal{N}^\rho_{\boldsymbol{p}})$ to the smooth function

$$E_\infty:\mathcal{N}_{\boldsymbol{p}}\to\mathbb{R},\;\;E_\infty(u)=\frac{1}{(2\pi|u|)^{\frac{m}{2}}}J_{\frac{m}{2}}(|u|),$$

where J_ν denotes the Bessel function of the first kind and order ν. □

Remark 1.1. (a) The limit function $E_\infty(u)$ has a more suggestive description, namely

$$E_\infty(u)=\frac{1}{(2\pi|u|)^{\frac{m}{2}}}J_{\frac{m}{2}}(|u|)=\frac{1}{(2\pi)^m}\int_{\boldsymbol{B}^m_1}e^{\boldsymbol{i}(\xi,u)}|d\xi|,\;\;\boldsymbol{B}^m_r:=\big\{\xi\in\mathbb{R}^m;\;\;|\xi|^2\leq r\big\}. \tag{1.1}$$

We denote by $\boldsymbol{E}_L$ the spectral function of the Laplacian on $\mathbb{R}^m$ corresponding to (generalized) eigenvalues $\leq L$. We then have (see [6, Eq. (2.1)] or [11, Eq.(1.3)])

$$\boldsymbol{E}_L(x,y)=\frac{1}{(2\pi)^m}\int_{\boldsymbol{B}^m_L}e^{\boldsymbol{i}(\xi,x-y)}|d\xi|.$$

This shows that

$$E_\infty(u)=\boldsymbol{E}_1(u,0).$$

(b) We ought to elaborate on the crux of the Universality conjecture. First of all, let us point out that known local Weyl trace formulæ ([2], [4, §17.5], [13, Thm. 1.8.5], [15, §7,8]) imply that the family $(\bar{\mathcal{E}}_L)_{L\geq1}$ is precompact in the C^∞-topology and as detailed in see Section 2, any limit point $\bar{\mathcal{E}}_\infty$ is asymptotically equivalent to E_∞ at $u=0$, i.e., for any $N>0$,

$$\bar{\mathcal{E}}_{\infty,\boldsymbol{p}}(u)=E_\infty(u)+O(|u|^{-N})\;\;\text{as}\;\;u\searrow0.$$

The Universality Conjecture makes the stronger claim that the family $(\bar{\mathcal{E}}_L)_{L\geq1}$ has a *unique* limit point in C^∞-topology and moreover, that limit point is *not just asymptotically equivalent, but equal* to the function E_∞.

(c) Recently, Lapointe, Polterovich and Safarov [7] have described a global type of relationship between $\mathcal{E}_L$ and $\boldsymbol{E}_L$ as $L\to\infty$.

□

The main result of this paper states that the Universality Conjecture is true in the case when M and g are real analytic. We achieve this in Section 2 by relying on some *sharp* a priori estimates that we prove in Appendix A, Theorem A.1. A weaker version of these estimates were used by Donnelly and Fefferman [3, §7] in their investigation of nodal sets of eigenfunctions on real analytic Riemannian manifolds. We believe that Theorem A.1 will find many other uses. In Section 3 we give an alternate proof of the conjecture in the special case when (M, g) is the round sphere.

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2. THE UNIVERSALITY CONJECTURE IN THE REAL ANALYTIC CASE

Theorem 2.1. *The Universality Conjecture is true when M and g are real analytic.*

Proof. Since both M and g are real analytic we deduce that the spectral function $\mathcal{E}_L$ is real analytic; see [8, 9]. The Cauchy-Kowaleskaya theorem, [5], implies that the exponential map is also real analytic so that $\mathcal{E}_L^\hbar$ is real analytic.

Fix a point $\boldsymbol{p}_0$ in M and normal coordinates $x = (x^1, \dots, x^m)$ at $\boldsymbol{p}_0$ defined on an open neighborhood $\mathcal{O}$ of $\boldsymbol{p}_0$. With these choices we can regard the restriction of $\mathcal{E}_L$ to $\mathcal{O} \times \mathcal{O}$ as a real analytic function $\mathcal{E}_L(x, y)$ defined on a neighborhood of $(0, 0)$ in $\mathbb{R}^m \times \mathbb{R}^m$. Via the exponential map

$$\exp_{(\boldsymbol{p}_0,\boldsymbol{p}_0)} : T_{(\boldsymbol{p}_0,\boldsymbol{p}_0)}M \times M \to M \times M$$

we can identify the space $\big\{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m;\;\; x + y = 0\big\}$ with the fiber $\mathcal{N}_{\boldsymbol{p}_0}$.

The results of [2] show that as $L \to \infty$ we have

$$\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}\mathcal{E}_L(x, y)_{x=y=0} = L^{\frac{m+|\alpha|+|\beta|}{2}}\Big(C_{\alpha,\beta} + O\big(L^{-\frac{1}{2}}\big)\Big), \tag{2.1}$$

where $C_{\alpha,\beta} = 0$ if $\alpha - \beta \notin 2\mathbb{Z}^m$, while if $\alpha - \beta \in 2\mathbb{Z}^m$ we have

$$C_{\alpha,\beta} = (-1)^{\frac{|\alpha|-|\beta|}{2}} \frac{1}{(4\pi)^{\frac{m}{2}} 2^{\frac{|\alpha|+|\beta|}{2}} \Gamma\big(1 + \frac{m}{2} + \frac{|\alpha|+|\beta|}{2}\big)} \prod_{i=1}^m (\alpha_i + \beta_i - 1)!!.$$

In fact we can say a bit more. More precisely, according to [4, Thm. 17.5.3], for any α, β there exists a constant $K_{\alpha,\beta} > 0$ that depends on the geometry of (M, g) but it is independent of $\boldsymbol{p}_0$ such that

$$\left|\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}\mathcal{E}_L(x, y)\right| \leq K_{\alpha,\beta} L^{\frac{m+|\alpha|+|\beta|}{2}}, \;\; \forall x, y. \tag{2.2}$$

A priori, the constants $K_{\alpha,\beta}$ can grow really fast as $|\alpha| + |\beta| \to \infty$. Our next result provides a key upper bound on this growth. To keep the flow of arguments uninterrupted we deferred its rather sneaky proof to Appendix A.

Lemma 2.2. *There exist constant $C, T > 0$ such that for any $L > 1$ and any multi-indices α, β we have*

$$\sup_{(\boldsymbol{p},\boldsymbol{q}) \in M \times M} |\partial^{\alpha+\beta}_{x,y}\mathcal{E}_L(\boldsymbol{p}, \boldsymbol{q})| \leq CT^{|\alpha|+|\beta|}\big(\,|\alpha| + |\beta|\,\big)!L^{\frac{m+|\alpha|+|\beta|}{2}}, \tag{2.3}$$

where (x, y) denote normal coordinates at $(\boldsymbol{p}, \boldsymbol{q})$. In other words, in (2.2) we can choose constants $K_{\alpha,\beta}$ satisfying

$$K_{\alpha,\beta} \leq KT^{|\alpha|+|\beta|}(|\alpha| + |\beta|)!, \tag{2.4}$$

where K and T are independent of α, β and L. □

Introduce new coordinates

$$\eta^i := \frac{1}{\sqrt{2}}(x^i - y^i),\;\; \tau^j := \frac{1}{\sqrt{2}}(x^j + y^j),\;\; i,j = 1,\dots,m.$$

The fiber $\mathcal{N}_{\boldsymbol{p}_0}$ is described by the equations $\tau^j = 0$, $j = 1,\dots,m$. Note that

$$x^j = \frac{1}{\sqrt{2}}(\eta^j + \tau^j),\;\; y^j = \frac{1}{\sqrt{2}}(\tau^j - \eta^j).$$

Along $\mathcal{N}_{\boldsymbol{p}_0}$ we can use the functions (η^j) as orthonormal coordinates. We have

$$r^2 = \sum_i (\eta^i)^2,$$

and

$$\mathcal{E}_{L,\boldsymbol{p}_0} = \mathcal{E}_L(2^{-\frac{1}{2}}\eta, -2^{-\frac{1}{2}}\eta),\;\; \bar{\mathcal{E}}_{L,\boldsymbol{p}_0}(\eta) = L^{-\frac{m}{2}}\mathcal{E}_L\left(\frac{1}{(2L)^{\frac{1}{2}}}\eta, -\frac{1}{(2L)^{\frac{1}{2}}}\eta\right)$$

We set

$$C_\alpha(L) := \frac{\partial^\alpha}{\partial^\alpha_\eta}\bar{\mathcal{E}}_{L,\boldsymbol{p}_0}(\eta)|_{\eta=0}.$$

From (2.4) we deduce that there exist constants $K, T > 0$ such that for any multi-index α and any $L > 1$ we have

$$|C_\alpha(L)| \leq K|\alpha|! T^{|\alpha|}. \tag{2.5}$$

Using (2.1) we deduce that

$$\partial^\ell_{\eta^i}\bar{\mathcal{E}}_L(x,y)_{x=y=0} = C_\ell + O\big(L^{-\frac{1}{2}}\big),$$

where $C_\ell = 0$ if ℓ is odd, while if ℓ is even we have

$$C_\ell = 2^{-\frac{\ell}{2}}\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}\partial^{\ell-k}_{x^i}\partial^k_{y^i}\bar{\mathcal{E}}_L(x,y)_{x=y=0} = (-1)^{\frac{\ell}{2}}\frac{(\ell-1)!!}{(4\pi)^{\frac{m}{2}}\Gamma(1+\frac{m}{2}+\frac{\ell}{2})}.$$

More generally,

$$C_\alpha(L) := O\big(L^{-\frac{1}{2}}\big) \;\text{ if }\; \alpha \notin 2\mathbb{Z}, \tag{2.6}$$

and

$$C_{2\alpha}(L) = (-1)^{|\alpha|}\frac{1}{(4\pi)^{\frac{m}{2}}\Gamma(1+\frac{m}{2}+|\alpha|)}\prod_{j=1}^m(2\alpha_j - 1)!! + O\big(L^{-\frac{1}{2}}\big). \tag{2.7}$$

The estimates (2.5), (2.6) and (2.7) show that there exists $\rho > 0$ such the family $\big\{\bar{\mathcal{E}}_L\big\}_{L\geq 1}$ of real analytic functions converges in the topology of $C^\infty(\{|\eta| < \rho\})$ to a function $\mathcal{E}_\infty$ that is real analytic on the ball $\{|\eta| < \rho\}$. Moreover,

$$\partial^{2\alpha}_\eta\mathcal{E}_\infty(0) = (-1)^{|\alpha|}\frac{1}{(4\pi)^{\frac{m}{2}}\Gamma(1+\frac{m}{2}+|\alpha|)}\prod_{j=1}^m(2\alpha_j-1)!!. \tag{2.8}$$

We deduce that for $|\eta| < \rho$ we have

$$\mathcal{E}_\infty(\eta) = \sum_\alpha\frac{1}{\alpha!}\partial^\alpha_\eta\mathcal{E}_\infty(0)\eta^\alpha = \sum_{n\geq 0}\frac{(-1)^n}{(4\pi)^{\frac{m}{2}}\Gamma(1+\frac{m}{2}+n)}\sum_{|\alpha|=n}\frac{\prod_j(2\alpha_j-1)!}{\prod_j(2\alpha_j)!}\eta^{2\alpha}$$

$$= \sum_{n\geq 0}\frac{(-1)^n}{(4\pi)^{\frac{m}{2}}\Gamma(1+\frac{m}{2}+n)}\sum_{|\alpha|=n}\frac{1}{2^{|\alpha|}\alpha!}\eta^{2\alpha} = \sum_{n\geq 0}\frac{(-1)^n}{(4\pi)^{\frac{m}{2}}2^n n!\Gamma(1+\frac{m}{2}+n)}\sum_{|\alpha|=n}\frac{n!}{\prod_j\alpha_j!}\eta^{2\alpha}$$

$$= \sum_{n\geq 0} \frac{(-1)^n}{(4\pi)^{\frac{m}{2}} 2^n n! \Gamma(1+\frac{m}{2}+n)} \Big(\sum_i (\eta^i)^2 \Big)^n = \sum_{n\geq 0} \frac{(-1)^n}{(4\pi)^{\frac{m}{2}} n! \Gamma(1+\frac{m}{2}+n)} \Big(\frac{r^2}{2} \Big)^n$$

$$= \frac{1}{(2\pi r)^{\frac{m}{2}}} \times \underbrace{\Big(\frac{r}{2} \Big)^{\frac{m}{2}} \sum_{n\geq 0} \frac{(-1)^n}{n! \Gamma(1+\frac{m}{2}+n)} \Big(\frac{r^2}{2} \Big)}_{=: J_{\frac{m}{2}}(r)}.$$

□

3. THE SPECTRAL FUNCTION OF A ROUND SPHERE

To appreciate the complexity involved in the Universality Conjecture we believe that it is instructive to give an alternate proof of the conjecture in the special case when M is the round sphere $S^{d-1} \subset \mathbb{R}^d$.

The spectrum of the Laplacian on S^{d-1} is (see [10])

$$\lambda_n = n(n+d-2), \;\; n = 0, 1, 2, \dots$$

and

$$\dim \ker(\lambda_n - \Delta) = \mu_n = \frac{2n+d-2}{n+d-2} \binom{n+d-2}{d-2}.$$

We set

$$\mathcal{H}_m := \ker(\lambda_m - \Delta), \;\; \boldsymbol{U}_n = \bigoplus_{k=0}^{n} \mathcal{H}_m.$$

As is well known, the space $\mathcal{H}_k$ coincides with the space of restrictions to S^{d-1} of harmonic polynomials of degree k in d variables. Fix an is an orthonormal basis $(\Psi_{k,m})_{1\leq\alpha\leq\mu_m}$ of $\mathcal{H}_m$ and set

$$\mathcal{E}_n(\boldsymbol{p}, \boldsymbol{q}) := \mathcal{E}_{\lambda_n}(\boldsymbol{p}, \boldsymbol{q}) = \sum_{m=0}^{n} \sum_{k=1}^{\mu_m} \Psi_{k,m}(\boldsymbol{p}) \Psi_{k,m}(\boldsymbol{q}).$$

The addition formula for spherical harmonics [10] implies that

$$\sum_{k=1}^{\mu_m} \Psi_{k,m}(\boldsymbol{p}) \Psi_{k,m}(\boldsymbol{q}) = \frac{\mu_m}{\boldsymbol{\sigma}_{d-1}} P_{m,d}(\boldsymbol{p} \bullet \boldsymbol{q}), \;\; \forall \boldsymbol{p}, \boldsymbol{q} \in S^{d-1}, \tag{3.1}$$

where $\boldsymbol{p} \bullet \boldsymbol{q}$ is the canonical inner product of $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^d$, $\boldsymbol{\sigma}_{d-1}$ is the area of S^{d-1},

$$\boldsymbol{\sigma}_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}, \tag{3.2}$$

and $P_{n,d}$ is the Legendre polynomial of degree n and order d,

$$P_{n,d}(t) := (-1)^n \frac{1}{2^n [n+\frac{d-3}{2}]_n} (1-t^2)^{-\frac{d-3}{2}} \frac{d^n}{dt^n} (1-t^2)^{n+\frac{d-3}{2}},$$

$$[x]_k := x(x-1)\cdots\big(x-(k-1)\big) = \frac{\Gamma(x+1)}{\Gamma(x+1-k)}.$$

The collection $(P_{n,d})_{n\geq 0}$ is an orthogonal family of polynomials with respect to the measure $w(t)dt = (1-t^2)^{\frac{d-3}{2}} dt$ on $[-1,1]$. More precisely we have, [10, §2],

$$\int_{-1}^{1} P_{n,d}(t) P_{m,d}(t) w(t) dt = h_n \delta_{n,m}, \;\; h_n = \frac{\boldsymbol{\sigma}_{d-1}}{\boldsymbol{\sigma}_{d-2} \mu_n}.$$

Observe that we can rephrase (3.1) as

$$\sum_{k=1}^{\mu_m} \Psi_{k,m}(\boldsymbol{p})\Psi_{k,m}(\boldsymbol{q}) = \frac{P_{m,d}(\boldsymbol{p}\bullet\boldsymbol{q})}{\boldsymbol{\sigma}_{d-2}h_m}. \tag{3.3}$$

We denote by k_n the leading coefficient of $P_{n,d}$. Its precise value is known, [10, Eq. (7.6)], but all we need in the sequel is the equality [10, Eq. (7.7)]

$$\frac{k_{n-1}}{k_n} = \frac{n+d-3}{2n+d-4}.$$

We recall the Christoffel-Darboux formula, [1, §5.2],

$$\sum_{m=0}^{n} \frac{P_{m,d}(s)P_{m,d}(t)}{h_m} = \frac{k_n}{k_{n+1}}\frac{P_{n+1,d}(t)P_{n,d}(s) - P_{n+1,d}(s)P_{n,d}(t)}{(t-s)h_n} \tag{3.4}$$

In (3.4) we let $s=1$. Using the equality $P_{m,d}(1)=1$, $\forall m$, we deduce

$$\sum_{m=0}^{n} \frac{P_{m,d}(t)}{h_m} = \frac{k_n}{k_{n+1}}\frac{P_{n+1,d}(t) - P_{n,d}(t)}{h_n(t-1)}. \tag{3.5}$$

Suming (3.3) for $m=0,\dots,n$ we deduce from (3.5) that

$$\mathcal{E}_n(\boldsymbol{p},\boldsymbol{q}) = \frac{k_n}{k_{n+1}}\frac{P_{n+1,d}(t) - P_{n,d}(t)}{\boldsymbol{\sigma}_{d-2}h_n(t-1)} = \frac{\mu_n}{\boldsymbol{\sigma}_{d-1}}\frac{k_n}{k_{n+1}}\frac{P_{n+1,d}(t) - P_{n,d}(t)}{(t-1)}.$$

Hence

$$\mathcal{E}_n(\boldsymbol{p},\boldsymbol{q}) = \mathcal{E}_n(\varphi) := \frac{\mu_n}{\boldsymbol{\sigma}_{d-1}}\frac{k_n}{k_{n+1}}\frac{P_{n+1,d}(\cos\varphi) - P_{n,d}(\cos\varphi)}{(\cos\varphi-1)},\quad \cos\varphi = \boldsymbol{p}\bullet\boldsymbol{q}. \tag{3.6}$$

Observe that φ is the geodesic distance between $\boldsymbol{p}$ and $\boldsymbol{q}$. Now set $r_n := \sqrt{\lambda_n}$ and define

$$\bar{\mathcal{E}}_n(\varphi) := r_n^{-(d-1)}\mathcal{E}_n\Big(\frac{\varphi}{r_n}\Big) = \frac{\mu_n}{\boldsymbol{\sigma}_{d-1}r_n^{d-2}}\frac{k_n}{k_{n+1}}\frac{P_{n+1,d}(\cos\frac{\varphi}{r_n}) - P_{n,d}(\cos\frac{\varphi}{r_n})}{r_n(\cos\frac{\varphi}{r_n}-1)}.$$

Taking into account Remark 1.1, we see that the Universality Conjecture will follow from the equality

$$\bar{\mathcal{E}}_n(\varphi) = \frac{1}{(2\pi\varphi)^{\frac{d-1}{2}}}J_{\frac{d-1}{2}}(\varphi) + O(n^{-1}),\ \text{ uniformly for }\ |\varphi|\leq\frac{\pi}{4}. \tag{3.7}$$

Observe first that

$$P_{n,d}(t) = \frac{1}{\binom{n+\alpha}{n}}P_n^{(\alpha,\alpha)}(t),\ \ \alpha = \frac{d-3}{2},\ \ \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}, \tag{3.8}$$

where $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials defined in [14, §4.1]. To prove (3.7) we will use the Hilb type asymptotic estimate for Jacobi polynomials., [14, Eq. (8.2.17)] or [12, (29)]. Here are the details. Set

$$\theta := \frac{\varphi}{r_n}$$

Observe that

$$\frac{1}{\binom{n+\alpha}{n}} = O(n^{-\alpha}). \tag{3.9}$$

The Hilb type estimate [14, Eq. (8.2.17)] coupled with (3.8) and (3.9) yields

$$\begin{aligned} P_{n,d}(\cos\theta) &= \Gamma(\alpha+1)\left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}}\frac{2^\alpha}{a_n^\alpha\sin^\alpha\theta}J_\alpha(a_n\theta)+\frac{\theta^{\alpha+2}}{\sin^\alpha\theta}O(1)\\ &= \Gamma(\alpha+1)\left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}}\frac{(2\theta)^\alpha}{\sin^\alpha\theta}\frac{1}{(a_n\theta)^\alpha}J_\alpha(a_n\theta)+\frac{\theta^{\alpha+2}}{\sin^\alpha\theta}O(1), \end{aligned}$$

where α is as in (3.8) and $a_n := n+\frac{d-2}{2}$. We set

$$F_\alpha(x) := \frac{1}{x^\alpha}J_\alpha(x)$$

and we deduce that

$$P_{n,d}(\cos\theta) = \Gamma(\alpha+1)\left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}}\frac{(2\theta)^\alpha}{\sin^\alpha\theta}F_\alpha(a_n\theta)+\frac{\theta^{\alpha+2}}{\sin^\alpha\theta}O(1).$$

Hence

$$\begin{aligned} \frac{P_{n+1,d}(\cos\theta)-P_{n,d}(\cos\theta)}{r_n(\cos\theta-1)} &= -\Gamma(\alpha+1)\frac{P_{n+1,d}(\cos\theta)-P_{n,d}(\cos\theta)}{2r_n\sin^2(\frac{\theta}{2})}\\ = -\Gamma(\alpha+1)\left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}}\frac{(2\theta)^\alpha}{\sin^\alpha\theta}&\Big(\frac{F_\alpha(a_{n+1}\theta)-F_\alpha(a_n\theta)}{2r_n\sin^2(\frac{\theta}{2})}\Big)+\frac{\theta^{\alpha+2}}{r_n\sin^{\alpha+2}\theta}O(1). \end{aligned}$$

Observe that $a_{n+1}\theta = a_n\theta+\theta$ and we deduce

$$F_\alpha(a_{n+1}\theta)-F_\alpha(a_n\theta) = F_\alpha'(a_n\theta)\theta+O(\theta)^2.$$

The classical identity involving Bessel functions, [1, Eq. (4.6.2)],

$$\frac{d}{dx}\Big(x^{-\nu}J_\nu(x)\Big) = -x^{-\nu}J_{\nu+1}(x),$$

implies that $F_\alpha'(z) = -x^{-\alpha}J_{\alpha+1}(x)$. Now observe that

$$2r_n\sin^2\Big(\frac{\theta}{2}\Big) = \frac{r_n\theta^2}{2}\big(1-O(\theta^2)\big).$$

Since

$$O(\theta) = O(r_n^{-1}) = O(n^{-1})$$

we deduce

$$\begin{aligned} \frac{P_{n+1,d}(\cos\theta)-P_{n,d}(\cos\theta)}{r_n(\cos\theta-1)} &= -\left(\frac{\theta}{\sin\theta}\right)^{\frac{1}{2}}\frac{2^{\alpha+1}\theta^\alpha}{\sin^\alpha\theta}\frac{F_\alpha'(a_n\theta)}{r_n\theta}+O(n^{-1})\\ &= -\frac{2^{\alpha+1}}{\varphi}F_\alpha'(\varphi)+O(n^{-1}) = \left(\frac{r}{2}\right)^{-(\alpha+1)}J_{\alpha+1}(r)+O(n^{-1}). \end{aligned}$$

Hence

$$\bar{\mathcal{E}}_n(\varphi) = \Gamma(\alpha+1)\frac{\mu_n}{\boldsymbol{\sigma}_{d-1}r_n^{d-2}}\frac{k_n}{k_{n+1}}\left(\frac{r}{2}\right)^{-(\alpha+1)}J_{\alpha+1}(r)+O(n^{-1}).$$

The equality (3.7) now follows from the estimates

$$\frac{\mu_n}{r_n^{d-2}} = \frac{2}{(d-2)!}+O(n^{-1}),\quad \frac{k_n}{k_{n+1}} = \frac{1}{2}+O(n^{-1}),$$

the equality (3.2), and the doubling formula

$$\sqrt{\pi}\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma\big(x+\frac{1}{2}\big).$$

□

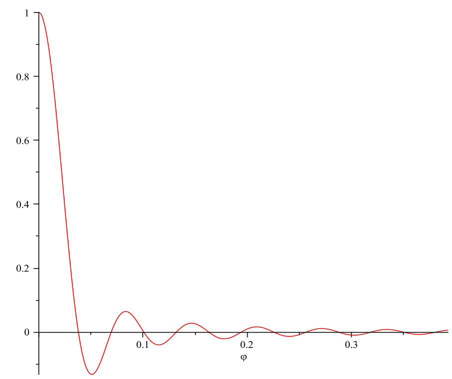

FIGURE 1. A depiction of $\frac{1}{\mathcal{E}_{100}(0)}\mathcal{E}_{100}(\varphi)$, $0 \leq \varphi \leq \frac{\pi}{8}$.

Remark 3.1. In the special case $M = S^2$ we can visualize the blowup behavior of the spectral function. In this case the eigenvalues of the Laplacian are $\lambda_n = n(n+1)$ and

$$\bar{\mathcal{E}}_n(\varphi) = \frac{1}{\lambda_n}\mathcal{E}_n\left(\frac{\varphi}{\sqrt{\lambda_n}}\right).$$

The function $\mathcal{E}_n(\varphi)$ has a peak at $\varphi = 0$,

$$\mathcal{E}_n(0) = \frac{(n+1)^2}{4\pi} \sim \frac{1}{4\pi}\lambda_n \text{ as } n \to \infty.$$

The oscillations in the graph depicted in Figure 1 are pushed at ∞ by the rescaling in φ, and the resulting behavior becomes rather tame, Figure 2.

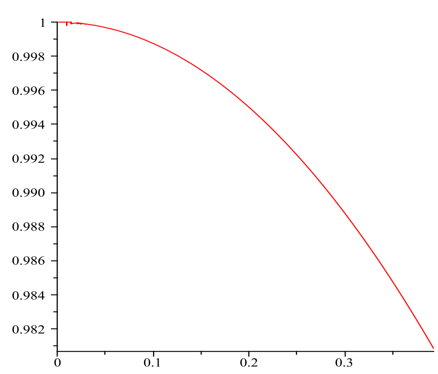

FIGURE 2. A depiction of $\frac{1}{\mathcal{E}_{100}(0)}\mathcal{E}_{100}(\varphi/100)$, $0 \leq \varphi \leq \frac{\pi}{8}$. The rescaling $\varphi \mapsto \frac{\varphi}{100}$ pushes the oscillations outside the screen.

□

APPENDIX A. SHARP ELLIPTIC ESTIMATES

The main goal of this appendix is to provide a proof of the estimates (2.3). We will follow the same strategy employed in the proof of [4, Thm. 17.5.3]. We will reach a conclusion stronger than [4, Thm. 17.5.3] because we will rely on *sharp* a priori elliptic estimates. We obtain these sharp estimates from the precise a priori estimates for elliptic operators with real analytic coefficients in [8, Chap. 8]. For the reader's convenience we first synchronize our notations with those in [8].

For any nonnegative integers s we set $M_s := s!$. Observe that [8, Chap. 8, Eq. (1.10)] becomes

$$M_{t+s} \leq d^{t+s} M_t M_s, \;\; \forall s,t \in \mathbb{Z}_{\geq 0}, \;\; d := 2. \tag{A.1}$$

Denote by Δ the Laplacian operator defined by a real analytic metric g on the ball B_4, where

$$B_R = \big\{ x \in \mathbb{R}^m; \;\; |x| < R \big\}.$$

For any $R > 0$ we denote by $\mathcal{A}(B_R)$ the space of functions real analytic on the ball B_R. We set

$$\|u\|_{k,R} = \sum_{|\alpha|=k} \left(\int_{B_R} |\partial_x^\alpha u(x)|^2 |dx| \right)^{\frac{1}{2}}. \tag{A.2}$$

Theorem A.1. *Then there exist constants $K, T > 0$, $\rho_0 \in (0,1)$ that depend only on the real analytic metric g and satisfying the following properties. If $u \in \mathcal{A}(B_2)$ and $Z_0, L_0 > 1$ are such that*

$$\|\Delta^k u\|_{0,2} \leq Z_0 L_0^k M_{2k}, \;\; \forall k \geq 0, \tag{A.3}$$

then

$$\|u\|_{j,\rho_0} \leq Z_0 \big(T\sqrt{L_0} \big)^j M_j, \;\; \forall j \geq 0. \tag{A.4}$$

$$|\partial_x^\alpha u(0)| \leq K Z_0 L_0^{\frac{m}{4}} \big(2T\sqrt{L_0} \big)^j M_j, \;\; \forall j \geq 0, \; \forall |\alpha| = j. \tag{A.5}$$

We want to emphasize that K and T are independent *of u, L_0, Z_0.*

Proof. We follow closely the approach in [8, Chap. 8, Sec. 2]. Fix a smooth function $\chi : \mathbb{R} \to \mathbb{R}$ such that

$$\chi(t) = \begin{cases} 1, & t \leq 0 \\ 0, & t \geq 1. \end{cases}$$

For $\rho, \delta > 0$ we define

$$\varphi_{\rho,\delta} : \mathbb{R}^m \to \mathbb{R}, \;\; \varphi_{\rho,\delta}(x) = \chi\left(\frac{|x| - \rho}{\delta} \right).$$

Note that

$$\operatorname{supp} \varphi_{\rho,\delta} \subset B_{\rho,\delta}, \;\; \varphi_{\rho,\delta} \equiv 1 \;\text{ on } B_\rho$$

and for all $p \in \mathbb{Z}_{>0}$ there exists $\gamma_p > 0$ such that

$$\big| D^p \varphi_{\rho,\delta}(x) \big| \leq \gamma_p \delta^{-p}, \;\; \forall x \in \mathbb{R}^m.$$

From [8, Chap. 8, Sec. 2, Thm. 2.1]. we deduce the following.

Lemma A.2. *There exist $\rho_1, C_1 > 0$, $\rho_1 < 1$, such that if $0 < \rho < \rho + \delta < \rho_1$ and any $u \in \mathcal{A}(B_2)$ we have*

$$\|u\|_{2,\rho} \leq C_1 \left(\|\varphi_{\rho,\delta} \Delta u\|_{0,\rho+\delta} + \sum_{\ell=0}^{2m-1} \frac{1}{\delta^{2m-\ell}} \|u\|_{\ell,\rho+\delta} \right). \tag{A.6}$$

□

Arguing as in the proof of [8, Chap. 8, Lemma 2.5] we obtain the following result.

Lemma A.3. *Suppose that $a \in \mathcal{A}(B_2)$ satisfies*

$$\sup_{X \in B_3} \sum_{|\alpha|=r} |\partial^\alpha a(x)| \leq L^r M_r, \ \ \forall r \geq 0.$$

Then for every $r, s \in \mathbb{Z}_{\geq 0}$, $0 < \rho < \rho + \delta < 1$ and $u \in \mathcal{A}(B_2)$ we have

$$\begin{gathered} \sum_{|\beta|=s} \sum_{|\alpha|=r} \| \partial^\beta [\varphi_{\rho,\delta}(a\partial^\alpha u - \partial^\alpha (au) \|_{0,\rho+\delta} \\ \leq C_s^* \sum_{\ell=0}^{s} \frac{1}{\delta^\ell} \sum_{t=0}^{s-\ell} L^{r+t} M_{r+t} \sum_{i=0}^{r-1} \frac{1}{L^i M_i} \|u\|_{s-\ell-t+i,\rho+\delta}, \end{gathered} \tag{A.7}$$

where C_s^* depends only on s. □

Using Lemma A.2 and A.3 we deduce as in [8, Chap. 8, Lemma 2.6] the following result.

Lemma A.4. *Suppose that ρ_1, C_1 are as in Lemma A.2. For any $\varepsilon > 0$ there exists $\gamma(\varepsilon) > 0$ that depends only on the metric and ε such that for any $u \in \mathcal{A}(B_2)$, $k > 0$ and $\rho < \rho + \delta < \rho_1$ we have*

$$\begin{gathered} \|u\|_{2k+2,\rho} \leq C_1^* \Bigg\{ \|\Delta u\|_{2k,\rho+\delta} + \varepsilon \|u\|_{2k+2,\rho+\delta} \\ M_{2k} \sum_{s=0}^{k=1} \frac{\gamma(\varepsilon)^{k-s}}{M_{2s}} \|u\|_{2s+2,\rho+\delta} + \frac{M_{2k}}{\delta^2} \sum_{s=-1}^{k-1} \frac{\gamma(\varepsilon)^{k-s-1}}{M_{2s+2}} \|u\|_{2s+2,\rho+\delta} \Bigg\}, \end{gathered} \tag{A.8}$$

where the constant C_1 depends only on the metric g. □

For every $\lambda, R > 0$ such that $R < \rho_1$ we set

$$\sigma^k(u, \lambda, R) := \frac{1}{M_{2k}\lambda^{k+1}} \sup_{R/2 \leq \rho < R} (R-\rho)^{2k+2} \|u\|_{2k+2,\rho}.$$

Using Lemma A.4 as in the proof of [8, Chap. 8, Lemma 2.7] we obtain the following result.

Lemma A.5. *There exists $\lambda_1 > 0$ such that for any $R < \frac{1}{2}\rho_1$, $\lambda \geq \lambda_1$, $k \geq 0$ and every $u \in \mathcal{A}(B_2)$ we have*

$$\sigma^k(u, \lambda, R) \leq \frac{M_{2k-2}}{4M_{2k}} \sigma^{k-1}(\Delta u, \lambda R) + \frac{1}{4} \sum_{s=-1}^{k-1} \sigma^s(u, \lambda, R). \tag{A.9}$$

□

An argument identical to the one used in the proof of [8, Chap. 8, Thm. 2.2] and based on Lemma A.5 shows that if $u \in \mathcal{A}(B_2)$ satisfies (A.3), then for λ, R as in Lemma A.5 we have

$$\sigma^k(u, \lambda, R) \leq \frac{M_{2k+2}}{M_{2k}} Z_0 (4L_0 + 2)^{k+1}. \tag{A.10}$$

If we let $\lambda = \lambda_1$ and $R_0 = \frac{1}{4}\rho_1$ in the above inequality we deduce

$$\frac{1}{\lambda_1^{k+1}} \sup_{R_0/2 \leq \rho < R_0} (R_0 - \rho)^{2k+2} \|u\|_{2k+2,\rho} \leq M_{2k+2} Z_0 (4L_0 + 2)^{k+1}$$

In particular, if $\rho_0 = 3R_0/4$ and $\mu := \sqrt{\lambda_1}$ we deduce

$$\|u\|_{2k+2,\rho_0} \leq c_0 M_{2k+2}\left(\frac{4\mu}{R_0}\right)^{2k+2}(4L_0+2)^{k+1} = Z_0 M_{2k+2}\zeta^{2k+2}, \;\; \zeta := \frac{4\mu}{R_0}\sqrt{4L_0+2}.$$

We can assume that $\zeta > 1$.

Using the interpolation inequality [8, Chap. 8, Eq. (2.7)] we deduce that

$$\|u\|_{2k+1,\rho_0} \leq \zeta^{-1}\|u\|_{2k+2,\rho_0} + c_m\zeta\|u\|_{2k,\rho_0},$$

where $c_m > 0$ is a constant that depends only on the dimension m. Hence

$$\|u\|_{2k+1,\rho_0} \leq Z_0 M_{2k+2}\zeta^{2k+1} + c_m Z_0 M_{2k}\zeta^{2k+1}$$

$$= Z_0 M_{2k+1}\zeta^{2k+1}\Big((2k+2) + \frac{c_m}{(2k+1)}\Big) \leq Z_0 M_{2k+1}\big(\,(2+c_m)\zeta\,\big)^{2k+1}.$$

We deduce that for any $k \geq 0$ we have

$$\|u\|_{k,\rho_0} \leq Z_0 M_k Z^k, \;\; Z = (2+c_m)\zeta. \tag{A.11}$$

This proves (A.4) with

$$T = \frac{Z}{\sqrt{L_0+1}}.$$

Using the Sobolev lemma [9, Thm. 3.5.1] we deduce[1] that there exists a constant $K_m > 0$ that depends only on m such that for any $v \in C^\infty(B_{r/2})$ we have

$$|v(0)| \leq \frac{K_m}{r^{\frac{m}{2}}}\left(\sum_{j=0}^{p-1}\frac{r^j}{j!}\|v\|_{j,r} + \frac{r^p}{(p-1)!}\|v\|_{p,\rho_0}\right), \;\; p = p_m := \left\lfloor\frac{m}{2}\right\rfloor + 1.$$

We deduce that for any multi-index α such that $|\alpha| = k$ and any r such that $r/2 < \rho_0$ we have

$$|\partial^\alpha u(0)| \leq \frac{K_m}{r^{\frac{m}{2}}}\left(\sum_{j=0}^{p-1}\frac{r^j}{j!}\|v\|_{j+k,\rho_0} + \frac{r^p}{(p-1)!}\|u\|_{p+k,\rho_0}\right)$$

$$\stackrel{(A.11)}{\leq} \frac{K_m Z_0}{r^{\frac{m}{2}}}\left(\sum_{j=0}^{p-1}\frac{r^j(k+j)!}{j!}Z^{k+j} + \frac{r^p(k+p)!}{(p-1)!}Z^{k+p}\right)$$

$$= \frac{K_m Z_0 k! Z^k}{r^{\frac{m}{2}}}\left(\sum_{j=0}^{p-1}\binom{k+j}{j}(rZ)^j + p\binom{k+p}{p}(rZ)^p\right).$$

Recall that $Z = T\sqrt{L_0+1}$. Choose r of the form

$$r = \frac{1}{B\sqrt{L_0+1}} < \frac{1}{B\sqrt{2}}.$$

where B is sufficiently large so that

$$\frac{r}{2} = \frac{1}{2B\sqrt{L_0+1}} < \frac{1}{2B\sqrt{2}} < \rho_0,$$

and

$$(rZ) = \frac{T}{B} \leq \frac{1}{2}.$$

---

[1] We have to warn the reader that the Sobolev norms $\|\nabla^j v\|_{2,R}$ defined in [9] do not coincide with the ones defined in (A.2) but they are equivalent. The constants implied by this equivalence of norms depend only on j and m.

$$
\begin{aligned}
|\partial^\alpha u(0)| &\leq K_m B^m Z_0 (L_0+1)^{\frac{m}{4}} Z^k k! \left(\sum_{j=0}^{p-1} \binom{k+j}{j} 2^{-j} + p \binom{k+p}{p} 2^{-p} \right) \\
&\leq p_m K_m B^m Z_0 (L_0+1)^{\frac{m}{4}} Z^k k! \underbrace{\sum_{j=0}^{\infty} \binom{k+j}{j} 2^{-j}}_{=2^{k+1}} \\
&= 2p_m K_m B^m Z_0 (L_0+1)^{\frac{m}{4}} (2T\sqrt{L_0+1})^k k!
\end{aligned}
$$

This proves (A.5) with $K = 2p_m K_m B^m$. □

Remark A.6. The above arguments extend with no changes to the case when the metric belongs to a Gevrey space, or more general to one of the spaces $\mathcal{D}_{M_k}(B_4)$ of [8], where the weights M_k are subject to the constraints (1.6)-(1.11) in [8, Chap.8]. □

Theorem A.1 has the following immediate consequence.

Corollary A.7. *Suppose N is a compact real analytic manifold of dimension n and g is a real analytic metric on N with injectivity radius $r(N)$. Denote by Δ_g the Laplace operator of the metric g and by $\mathcal{A}(N)$ the space of real analytic functions on N. Then there exist constants $K, T > 0$ and $0 < \rho_0 < r(N)$ depending only on g with the following property: for any $Z_0, L_0 > 1$ and any $u \in \mathcal{A}(N)$ such that, if*

$$
\|\Delta_g^k u\|_{L^2(N,g)} \leq Z_0 L_0^k k!, \;\; \forall k \geq 0, \tag{A.12}
$$

then

$$
|\partial_x^\alpha u(\boldsymbol{p})| \leq K Z_0 L_0^{\frac{m}{4}} \big(2T\sqrt{L_0} \big)^j j!, \;\; \forall |\alpha| = j, \;\; \forall \boldsymbol{p} \in M, \tag{A.13}
$$

where $x = (x^1, \dots, x^m)$ are normal coordinates at $\boldsymbol{p}$. □

Proof of (2.3). We follow the same strategy used in the proof of [4, Thm. 17.5.3]. Fix $L > 1$ and denote by P_L the orthogonal projection onto the space

$$
H_L := \bigoplus_{\lambda \leq L} \ker(\lambda - \Delta).
$$

Fix $j, \ell \geq 0$ points $\boldsymbol{p}_0, \boldsymbol{q}_0$ in M, and normal coordinates x at $\boldsymbol{p}_0$ and y at $\boldsymbol{q}_0$. Let $f \in L^2(M)$ and set $f_L = P_L f$. Then

$$
\|\Delta^k f_L\|_{L^2(M)} \leq L^k \|f\|, \;\; \forall k \geq 0.
$$

Using Corollary A.7 we deduce that

$$
|\partial_x^\alpha f_L(\boldsymbol{p}_0)| \leq K L^{\frac{m}{4} + \frac{j}{2}} (2T)^j j! \|f\|, \;\; \forall \boldsymbol{p} \in M, \;\; |\alpha| = j,
$$

where K, T are independent of f, k, α and L. Now observe that

$$
\partial_x^\alpha f_L(\boldsymbol{p}_0) = (f, g_0)_{L^2(M)}, \;\; g_0(\boldsymbol{q}) = \partial_x^\alpha \mathcal{E}_L(\boldsymbol{p}_0, \boldsymbol{q}) = \sum_{\lambda_n \leq L} \big(\partial_x^\alpha \Psi_n(\boldsymbol{p}_0) \big) \Psi_n(\boldsymbol{q}).
$$

The above discussion shows that for every $f \in L^2(M)$ we have

$$
\big| (f, g_0)_{L^2(M)} \big| \leq K L^{\frac{m}{4} + \frac{j}{2}} (2T)^j j! \|f\|,
$$

so that

$$
\|g_0\|_{L^2(M)} \leq K L^{\frac{m}{4} + \frac{j}{2}} (2T)^j j!.
$$

Observe that $g_0 \in H_L$ and thus for any $k \geq 0$ we have

$$\|\Delta^k g_0\|_{L^2(M)} \leq L^k \|g_0\|_{L^2(M)}$$

and Corollary A.7 implies that for any $|\beta| = \ell$ we have

$$|\partial_x^\alpha \partial_y^\beta \mathcal{E}_L(\boldsymbol{p}_0, \boldsymbol{q}_0)| = |\partial_y^\beta g_0(\boldsymbol{q}_0)| \leq K L^{\frac{m}{4}+\frac{\ell}{2}} (4T)^\ell \ell! \|g\|_0 \leq K^2 L^{\frac{m}{2}+\frac{j}{2}+\frac{\ell}{2}} T^{j+\ell} j!\ell!$$

The inequality (2.3) follows by observing that

$$j!\ell! \leq \binom{j+\ell}{j} (j+\ell)! \leq 2^{j+\ell}(j+\ell)!.$$

□

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