New Developments in Mathematics

Proceedings of a Conference in honor Of the twentyfifth Birthday of Steven Spallone

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Contents

$\mathbb{Z}/2$ Analysis

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Abstract

We mimic the development of real analysis, replacing the real line with the field $\mathbb{Z}/2$.

1 Functions and derivatives

Let $\mathbb{Z}/2$ be the field with two elements. By a *function*, we mean a map of sets

$$f : \mathbb{Z}/2 \to \mathbb{Z}/2$$

and we shall denote the collection of all such maps $\mathcal{F}$.

Given $f \in \mathcal{F}$, and $a \in \mathbb{Z}/2$, we define the derivative of f at a to be

$$\frac{df}{dx}(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

One must interpret this very carefully on $\mathbb{Z}/2$. We recall from calculus that the limit is evaluated at x close to, but not equal to a. Therefore the limit of the expression is the expression evaluated at the unique point $x \in \mathbb{Z}/2$ not equal to a. It is clear that the denominator is non-zero for such x, hence the derivative always exists, and every function is differentiable.

Theorem.

$$\frac{df}{dx}(a) = \begin{cases} 1 & f \text{ non-constant} \\ 0 & f \text{ constant} \end{cases}$$

This follows immediately from the definition of the derivative. As a corollary, we are able to obtain a complete classification of functions.

Corollary. *Every function $f(x)$ is of the form $\alpha + \beta x$, where $\alpha, \beta \in \mathbb{Z}/2$.*

Proof. Given $f(x)$, if it is constant, $f(x) = \alpha$ for some $\alpha \in \mathbb{Z}/2$. Assume that f is non-constant. Then

$$\frac{d}{dx}(f(x) - x) = 1 - 1 = 0$$

by the previous theorem. Therefore, again by the previous theorem, we can conclude that $f(x) - x$ is constant, so $f(x) = \alpha + x$ for some $\alpha \in \mathbb{Z}/2$. □

Note that in fact, as a ring (under pointwise multiplication of functions) we have $\mathcal{F} = \mathbb{Z}/2[x]/(x^2 = x)$.

The first non-trivial result of $\mathbb{Z}/2$ analysis is the fact that the derivative, while being linear, does not obey a product rule. This is seen by observing that

$$\frac{d}{dx}x^3 \neq 3x^2 \; (= x)$$

but is in fact the constant function 1. It is straightforward, but tedious to verify that the proper product rule is

$$\left.\frac{d}{dx}\right|_a f(x)g(x) = \lim_{x \to a} \frac{f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)}{x - a}$$
$$= f(a+1)g'(a) + g(a)f'(a) \quad (\text{setting } x = a+1)$$

2 Measure and integration

A $\mathbb{Z}/2$-valued measure on $\mathbb{Z}/2$ is a function

$$\mu : \mathcal{P}(\mathbb{Z}/2) \to \mathbb{Z}/2$$

(where $\mathcal{P}(\mathbb{Z}/2)$ is the power set of $\mathbb{Z}/2$) satisfying the following two conditions.

1. $\mu(\emptyset) = 0$
2. $\mu(X \amalg Y) = \mu(X) + \mu(Y)$

where $\amalg$ denotes disjoint union. Denote the set of all $\mathbb{Z}/2$ valued measures on $\mathbb{Z}/2$ by $Meas(\mathbb{Z}/2)$. It is interesting to note that there is a natural correspondence $Meas(\mathbb{Z}/2) \cong \mathcal{F}$ obtained by associating to a measure μ a function f_μ where $f_\mu(a) = \mu(\{a\})$. In particular, $Meas(\mathbb{Z}/2)$ has the same number of elements as $\mathcal{F}$.

Define, for a subset X of $\mathbb{Z}/2$, a function f, and a measure μ, the integral

$$\int_X f(x)d\mu = \sum_{a \in X} f(a)\mu(a)$$

3 Results

We now state and prove our deepest results. The first is a Riesz representation theorem.

Theorem. *The map*

$$Meas(\mathbb{Z}/2) \to \mathcal{F}$$

obtained by sending a measure μ to the linear functional L_μ defined by

$$L_\mu(f) = \int_{\mathbb{Z}/2} f d\mu$$

is an isomorphism of $\mathbb{Z}/2$-vector spaces.

Proof. Clearly the correspondence is linear. Since $Meas(\mathbb{Z}/2)$ has the same dimension as $\mathcal{F}$, we need only show the map is injective. But if $\int f d\mu = 0$ for every function f, then in particular this is true for $f = \chi_{\{a\}}$, the characteristic function on the singleton $\{a\}$. But

$$\int \chi_{\{a\}} d\mu = \mu(\{a\})$$

so we conclude that μ was the zero measure. Thus our homomorphism has no kernel. □

We close by the main computational tool for $\mathbb{Z}/2$ integral calculus with respect to Lebesgue measure. This measure dx is uniquely characterized by

1. $dx(X) = dx(X + a)$ (Haar measure)
2. $dx(\{0\}) = 1$ (Normalization)

We prove a $\mathbb{Z}/2$ fundemental theorem of calculus.

Theorem.

$$\frac{d}{dx}\int_0^x f(y)dy = f(1)$$

Proof. For we calculate

$$F(x) = \int_0^x f(y)dy = f(0) + \begin{cases} 0 & x = 0 \\ f(1) & x = 1 \end{cases}$$

so the function $F(x)$ is non-constant if $f(1) = 1$ and is constant if $f(1) = 0$. The result follows from the first theorem of this paper. □

EXISTENCE AND APPLICATIONS OF THE HAPPY CATEGORY

SAMEER X. D'COSTA AND CATHERINE D. LEIGH

Abstract. We will prove the existence of the Happy Category ☺. Given any category $\mathfrak{P}^*$, the existence of a "free association" functor $\mathcal{F}$ from $\mathfrak{P}^*$ to ☺ translates all problems in $\mathfrak{P}^*$ to computations in the Happy Category. These computations are joyful, radiant, pleasing, lighthearted, gratifying, jovial, and as exhilarating as listening to Stephen Spallone sing. We shall demonstrate a singularly pulchritudinous application.

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(1) The research work done was made possible in part by GAAN Fellowship Number #AZ01253 (2000-01) and the Robert McCormick Fellowships.

(2) Both authors wish to state that although they do not endorse Deriving Under the Influence, much of this work would not have been done if they had stayed sober all the time.

(3) The first author would like to apologize for his inability to do anything productive with LaTeX.

We wish to show some applications of the Happy Category. This new branch of mathematics has tremendous applications in many old branches of mathematics (which are described using the language of categories). Hopefully it will be used to develop new branches of math which will have a profound impact on the way math is done in the next century. (For existence and uniqueness of the Happy Category please see our previous paper in the Annals of Mathematics Vol. XIXCMIV.)

The Happy category is basically a collection of "nice" objects and "nice" maps. We will prove

Theorem. *A matrix algebra of a division algebra is a simple algebra.*

as an example of this powerful technique. For this we use the "free association functor." (The definitions are all contained in the aforementioned paper — Ed.)

Proof. To prove that a matrix algebra of a division algebra is a simple algebra, free associating, we see that we just need to prove

$$matrix = matrix\ algebra \times \frac{1}{algebra} = simple$$

so we just have to prove that the matrix is simple. Suppose not. Then the matrix is complex. From the movie we learned that things that seem real in the real world are actually imaginary. So this means that real symmetric matrices have imaginary eigenvalues. This is not true. So the matrix is simple. So the matrix algebra is a simple algebra. □

Corollary. *The direct sum of matrix algebras over a division ring is semi-simple.*

Proof. Each component of the direct sum is simple. So a bunch of components collectively are not so simple, i.e. semi-simple. □

These techniques are going to revolutionize mathematics. Thanks to Rochelle and Pallavi for helping.

Quantum Taxicab Geometry

Brian Johnson
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We briefly describe the quantum taxi, followed by an introduction to the resulting geometry and its relationship to "ordinary" quantum geometry. We conclude with a brief discussion of conics in the quantum taxicab plane.

1 The quantum taxicab

Figure 1 illustrates the myrid collection of obstructions that impede the common taxicab from travelling from point A to point B, and these have been well studied in the literature, or at least have been folklore in taxidermy.

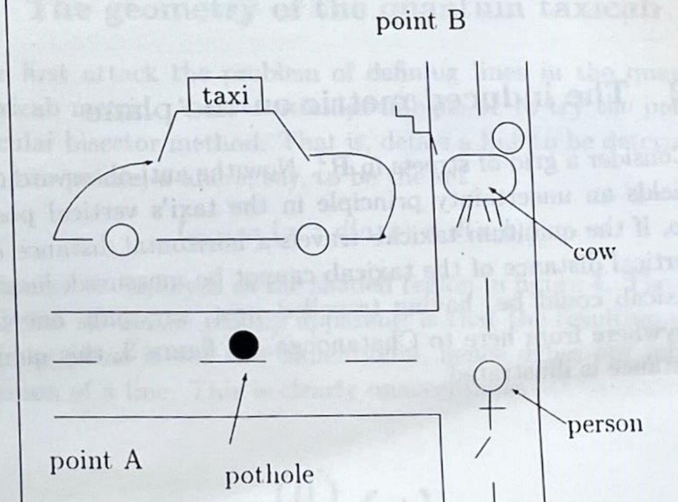


Figure 1: the Newtonian taxi

Our technique is to replace the classical taxi with a quantum taxi. Roughly, the quantization is achieved by replaceing the air conditioner (A.C.) with a complicated contruction known as the Anti-Observation Generator (A.O.G). It should be clear to the reader that this new quantized taxi, while a bit stuffy in the summer months, has far nicer properties then its classical counterpart. The contruction of supertaxis is still an open problem. Figure 2 illustrates the construction of the quantum taxi.

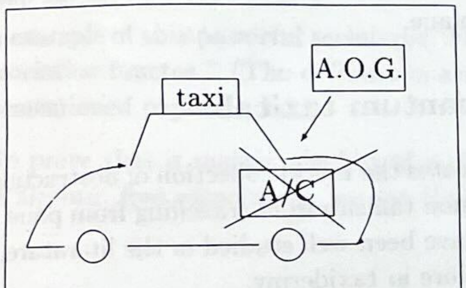


Figure 2: quantization of the classical taxi

2 The induced metric on the plane

Consider a grid of streets in $\mathbf{R}^2$. Now the anti-observaton effect yields an uncertainty principle in the taxi's vertical position. So, if the quantum taxicab travels a horizontal distance d, the vertical distance of the taxicab cannot be measured, hence the taxicab could be, having travelled west, say, only one block, anywhere from here to Chatanooga. In figure 3, this quantum distance is illustrated.

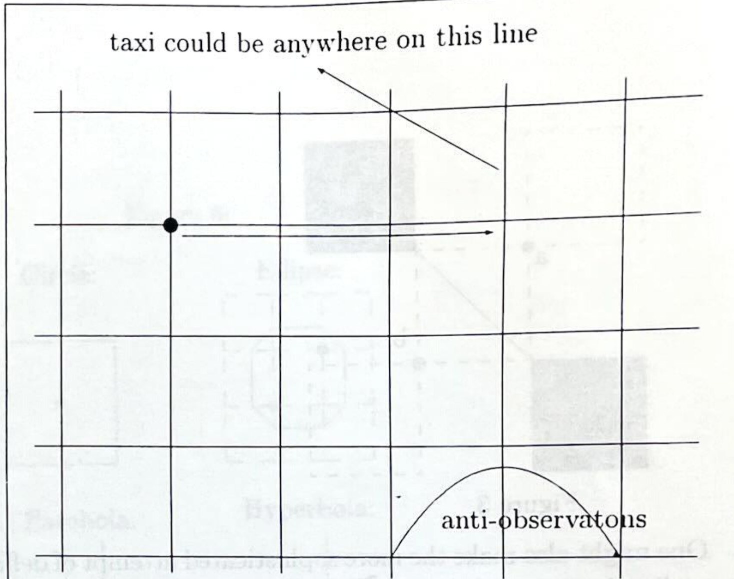


Figure 3: a quantum taxicab geodesic

3 The geometry of the quantum taxicab

We first attack the problem of defining lines in the quantum taxicab metric. A naive attempt mibght be to try the perpendicular bisector method. That is, define a line to be determined by two points, a and b, say, to be the set

$$L_{(a,b)} = \{x \ : \ d(a,x) = d(x,b)\}.$$

This set is displayed as the shaded region in figure 4. The problem, as should be readily apparent, is that the resulting subset of the plane is not one dimentional, hence defies our intuitive notion of a line. This is clearly unacceptable.

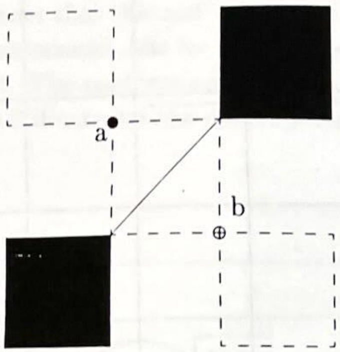


Figure 3

One might also make the more sophisticated attempt of defining a line in terms of shortest distance. Given points a and b, define the line segment from a to b to be the set

$$L_a^b = \{x \;:\; d(a,x) + d(x,b) = d(a,b)\}$$

The problem, again, is with the uncertainty inherent in the quantum taxicab. This line segment is illustrated as the shaded region in figure 5. We are led to believe that lines don't work in the quantum taxicab plane..

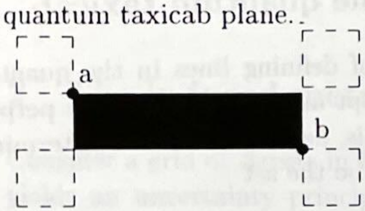
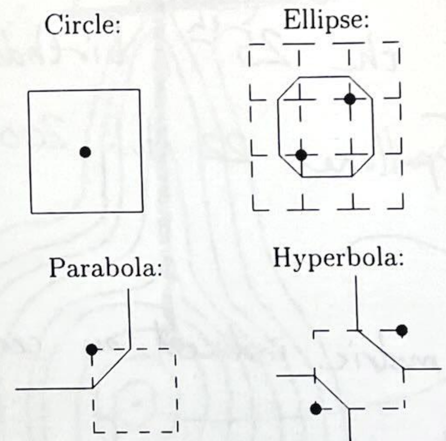


Figure 5

Interestingly enough, though, the quantum taxicab is not perverse enough to exclude conic sections of hausdorff dimension 1. These are illustrated in figure 6. It is interesting to note that each branch of the hyperbola is congruent to a parabola.

Figure 6:



Recall that a geometry is said to satisfy the Karl-Dieter Crisma (KDC) property if every pair of intersecting circles in general position intersect at right angles. The main result is the following, whose proof follows from the form that cicles in the quantum taxicab geometry take.

Theorem The quantum taxicab geometry satisfied the KDC property.

PUBLIC TRANSPORTATION

METRICS

Dan Margalit

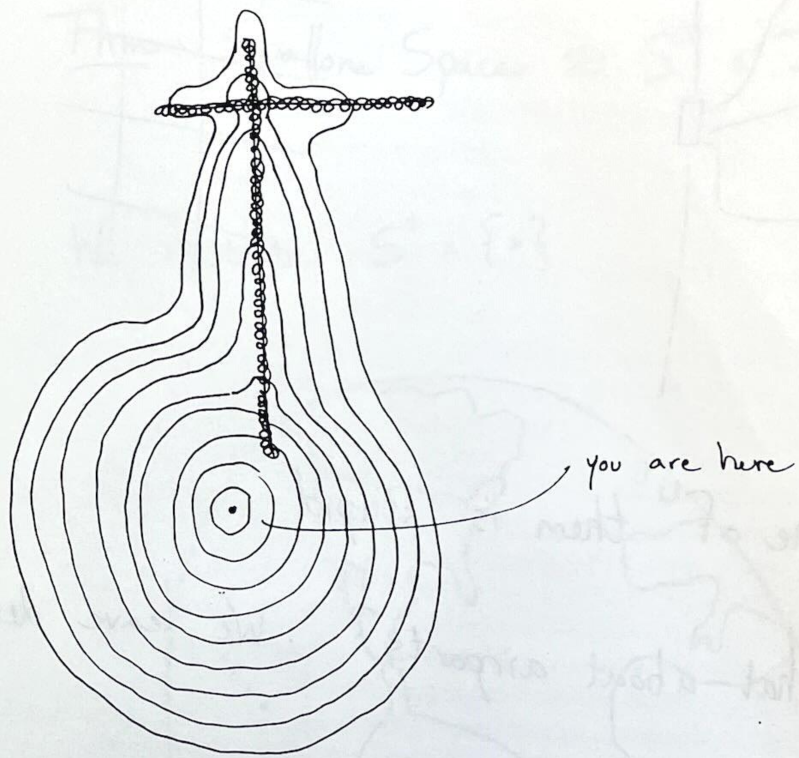
In honor of the 25^{th} birthday

of Steven Spallone 22 July 2001.

We define a metric induced on certain subsets of S^2 by various forms of public transportation. All definitions are what you think they are, unless you're stupid.

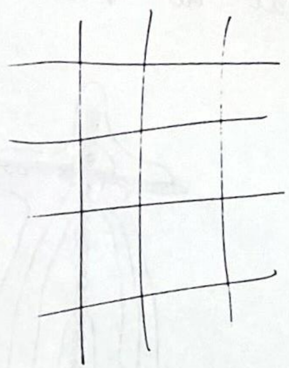
Consider a subway line. No-two.

What do metric balls look like?



Note that we have considered the Quantum Wait Effect at transfer points.

nsider two types of subway systems:



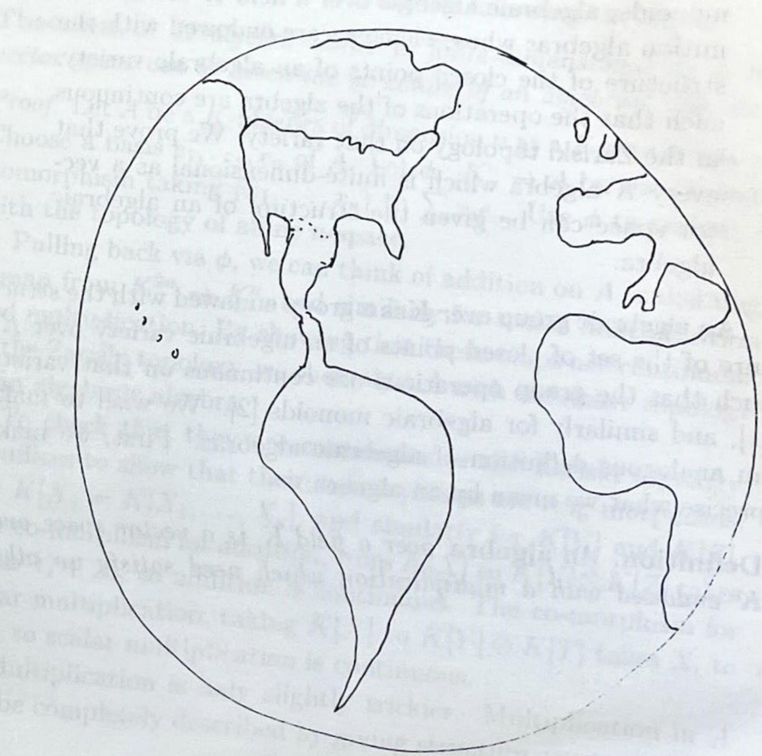
One of them is stupid.

What about airports? We leave details
o the reader.

We now define Spallone Space, which is obviously the space of all public ransportation metrics.

Thm: Spallone Space $\simeq S^2 \times I$

We illustrate $S^2 \times \{*\}$:



Algebraic Algebras

Hugh Thomas

dedicated to Steven Spallone on the occasion of his twenty-fifth birthday

Abstract

By analogy with algebraic groups and algebraic monoids, algebraic algebras over a field K are by definition algebras whose elements are endowed with the structure of the closed points of an algebraic variety, such that the operations of the algebra are continuous in the Zariski topology on that variety. We prove that every K-algebra which is finite-dimensional as a vector space can be given the structure of an algebraic algebra.

An algebraic group over K is a group endowed with the structure of the set of closed points of an algebraic variety over K, such that the group operations are continuous on that variety [1], and similarly for algebraic monoids [2]. We wish to make an analogous definition of algebraic algebras. First, we make precise what we mean by an algebra:

Definition. *An* **algebra** *over a field K is a vector space over K endowed with a multiplication which need satisfy no other requirements than distributativity over addition and scalar multiplication.*

Now, we can make the following definition:

Definition. *An* **algebraic algebra** *over K is an algebra over K, whose elements are endowed with the structure of the set of closed points of an algebraic variety over K, such that all the algebra operations are continuous in the Zariski topology on that variety.*

This note is devoted to proving the following theorem

Theorem. *A K-algebra which is finite-dimensional as a K-vector space can be given the structure of an algebraic algebra.*

Proof. Let A be a K-algebra of dimension n as a K-vector space. Choose a basis $e_1, \ldots, e_n$ of A. Let $\phi : K^n \to A$ be the obvious isomorphism taking $(x_1, \ldots, x_n)$ to $\sum x_i e_i$. Use ϕ to endow A with the topology of affine n-space.

Pulling back via ϕ, we can think of addition on A as defining a map from $K^{2n} \to K^n$, and similarly for scalar multiplication and multiplication. By showing that these maps are continuous in the Zariski topology, we show that A with the above topology is an algebraic algebra.

To check that they are continuous in the Zariski topology, it suffices to show that their co-morphisms are ring morphisms. Let $K[X] = K[X_1, \ldots, X_n]$, and similarly for $K[Y]$ and $K[Z]$. The co-morphism for addition, from $K[X]$ to $K[Y] \otimes K[Z]$ takes X_i to $Y_i + Z_i$, so addition is continuous. The co-morphism for scalar multiplication, taking $K[X]$ to $K[Y] \otimes K[T]$ takes X_i to TY_i, so scalar multiplication is continuous.

Multiplication is only slightly trickier. Multiplication in A can be completely described by giving structure constants $\Gamma^k_{ij} \in$

K, the coefficient of e_k in $e_i e_j$. Then the comorphism for multiplication, taking $K[X]$ to $K[Y] \otimes K[Z]$ takes $X_k \to \sum \Gamma^k_{ij} Y_i Z_j$, which is a ring map, so multiplication is also continuous, and A has been given the structure of an algebraic algebra. □

Note that this theorem leaves open the possibility that there could exist non-standard realizations of certain K-algebras as algebraic algebras, in which the K-algebra, though abstractly isomorphic to some affine space, would have a different topology.

References

[1] J. Humphreys, Linear Algebraic Groups, Springer-Verlag, New York, 1975.

[2] M. Putcha, Linear algebraic monoids, Cambridge University Press, Cambridge, 1988.

The two point compactification of topological spaces

Kevin Wortman

Recall the standard definition:

The one point compactification of a noncompact topological space X is the set $\dot{X} = X \cup \{\infty\}$ with the top

$$G \in \tau(\dot{X}) \iff \begin{cases} G \in \tau(X) \\ \text{or} \\ G = \dot{X} \setminus C \quad \text{for a compact } C \subseteq X \end{cases}$$

properties of $\dot{X}$:

① $\dot{X}$ is compact

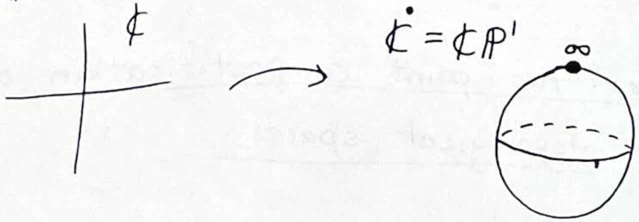
② $X \hookrightarrow \dot{X}$ with $\overline{X} = \dot{X}$

③ $\dot{X}$ may have a far more complicated homotopy type than that of X.

④ If X is T_2 and locally compact then $\dot{X}$ is T_2

Examples: Two most common are $\mathbb{C}P^1$ and S^3.

1 $\mathbb{C}P^1$



2 S^3

K a knot in $\mathbb{R}^3$.

"surgery": any closed M^3 can be gotten by taking some K's in $\mathbb{R}^3$, removing them and gluing them back in.

need this to be compact so pretend $K \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$.

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nfortunately, properties 3 and 4 are undesireable

: A contractible space shouldn't be converted into a space with positive cohomological dimension.

T_2 is a dinosaur assumption that prevents us from studying some of mathematics most interesting objects, such as Spec $\mathbb{Z}[x]$ which is only T_0.

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al: To produce an alternative compactification of topological spaces with the convenience of properties 1 and 2 but without the drawbacks of properties 3 and 4.

Definition: For a noncompact space X, the two point compactification of X is the set $\dot{\ddot{X}} = X \cup \{\infty_N, \infty_P\}$ with the topology

$$G \in \mathcal{T}(\dot{\ddot{X}}) \iff G \in \mathcal{T}(X)$$

or

$$G = \dot{\ddot{X}} \setminus C \quad \text{for a compact } C \subseteq X$$

or

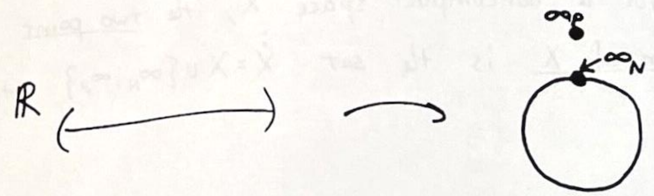
$$G = \dot{\ddot{X}} \setminus (C \cup \{\infty_P\}) \quad \text{for a compact } C$$

---

properties of $\dot{\ddot{X}}$:

1. $\dot{\ddot{X}}$ is compact
2. $X \hookrightarrow \dot{\ddot{X}}$ with $\overline{X} = \dot{\ddot{X}}$
3. $\pi_n(\dot{\ddot{X}}, \infty_P) = 0$ for $n > 0$
4. If X is T_0, then $\dot{\ddot{X}}$ is T_0

Example: $\ddot{\mathbb{R}}$



Open sets either are open sets in S^1 (ignoring ∞_P), or are open sets in S^1 which contain ∞_N unioned with $\{\infty_P\}$. Note: Clearly not T_1 since ∞_P can't be seperated from ∞_N.