Two-Way Analysis of Variance

> **Note:** Much of the math here is tedious but straightforward. We'll skim over it in class but you should be sure to ask questions if you don't understand it.

I.   OVERVIEW.

A.   Sometimes a researcher might want to simultaneously examine the effects of two treatments (where both treatments have nominal-level measurement).

EXAMPLES:

✓ the effect of sex and race on wages
✓ the effects of the level of pollution and the level of city services on housing prices
✓ the effects of religion and region on income

To elaborate: with sex and race, we might wonder if
✓ are there differences because of sex alone
✓ are there differences because of race alone
✓ are there differences attributable to particular combinations of sex and race - that is, are there interaction effects? For example, white males, white females, and black males may all have similar wages, but black females could have much lower wages. We'll discuss interaction effects more shortly.

B.   **Two-Way Anova with a Balanced Design and the Classic Experimental Approach**. We can use Analysis of Variance techniques for these and more complicated problems. These techniques can get fairly involved and employ several different options, each of which has various strengths and weaknesses. If this were a psychology class, we might spend a lot more time going over ANOVA, where such techniques are more widely used. But, in Sociology, we are much more likely to use regression and other techniques for our advanced work. Therefore, for our purposes, I will primarily focus on the special case of *balanced designs* (this is also what Hays, Harnett and other texts focus on). In a balanced design, *all cell frequencies are equal*, i.e. the number of observations in each combination of treatments is the same. So, for example, there would be 5 white males, 5 black males, 5 white females, and 5 black females. Balanced designs are unlikely in survey research but they are quite common (and often encouraged) in experimental studies. Equal cell frequencies make it easier to disentangle the effects of the row and column variables (e.g. sex and race) and also minimizes the effect of non-homogenous population variances if they exist.

In addition, I'll note that several programs give you various options for the "Method" to use for Anova. If the design is balanced, I don't think it matters what method you use. But, if you choose what SPSS calls the *Classic Experimental Approach*, many of the formulas that follow will be valid even when the design is not balanced. The *Regression Approach* and the *Hierarchical Approach* are other options (and several other options, with varying names, are also listed in different procedures). The SPSS manual and other sources have more information if you find yourself needing to know about these.

As noted below, these assumptions are not required for everything we will be talking about. These assumptions will affect how computations are done with the raw data but, once that is done, the hypothesis testing procedures will be largely the same. Ergo, the most critical parts of our discussion will apply even when designs are not balanced.

C.   The model. When we have 2 treatments, the model can be written as

$$y_{ijk} = \mu + \tau_j + \lambda_k + (\tau\lambda)_{jk} + \varepsilon_{ijk}$$

where μ = the grand mean, τ_j is the treatment effect for the jth category of the row variable, λ_k is the treatment effect for the kth category of the column variable, $(\tau\lambda)_{jk}$ is the interaction effect for the combination of the jth row category and the kth column category.
       EXAMPLE: Suppose the overall average income is $20,000, the average black income is $15,000, the average female income is $17,000, and the average black woman's income is $10,000. This means that μ = $20,000, τ_B = -$5,000, λ_W = -$3,000, $(\tau\lambda)_{BW}$ = -$2,000.

D.   As before, we want to partition the variance. Note that

$$s_y^2 = \frac{\Sigma\Sigma\Sigma(y_{ijk} - \bar{y})^2}{N-1} = \frac{Total\ SS}{N-1} = \frac{TSS}{N-1} = MS\ Total$$

Further, note that

Component	Description
$(y_{ijk} - \bar{y}) =$	Deviation of the individual score from the overall mean
$(y_{ijk} - \bar{y}_{jk})$	Deviation of the individual score from the group mean, i.e. $\hat{\varepsilon}_{ijk}$
$+(\bar{y}_j - \bar{y})$	Deviation of the jth row's mean from the overall mean, i.e. $\hat{\tau}_j$
$+(\bar{y}_k - \bar{y})$	Deviation of the kth column's mean from the overall mean, i.e. $\hat{\lambda}_k$
$+(\bar{y}_{jk} - \bar{y}_j - \bar{y}_k + \bar{y})$	Deviation of "combination" mean from row and column means; the interaction, i.e. $(\hat{\tau}\hat{\lambda})_{jk}$

Note that we are using the same trick we did before of adding and then subtracting the same terms.
       Hence, $\Sigma\Sigma\Sigma(y_{ijk} - \bar{y})^2$ can be broken out as follows (any seemingly omitted terms conveniently work out to be zero):

$$\sum\sum\sum(\,y_{ijk} - \overline{y}_{jk}\,)^2 = \sum\sum\sum \hat{\varepsilon}_{ijk}^2 = SS\ Error,$$

$$d.f. = N - JK$$

This is analogous to SS Within from 1-way ANOVA. This represents the deviation of individuals from the means of others who have the same value on the row and column variables (e.g. are of the same sex and race); that is, this represents the component of the scores that cannot be accounted for by group membership. The d.f. arise from the fact that there are N cases, and J*K means have to be estimated. Also,

$$\sum\sum\sum(\,\overline{y}_j - \overline{y}\,)^2 = \sum\sum\sum \hat{\tau}_j^2 = SS\ Rows,$$

$$d.f. = J - 1$$

$$\sum\sum\sum(\,\overline{y}_k - \overline{y}\,)^2 = \sum\sum\sum \hat{\lambda}_k^2 = SS\ Columns,$$

$$d.f. = K - 1$$

$$\sum\sum\sum(\,\overline{y}_{jk} - \overline{y}_j - \overline{y}_k + \overline{y}\,)^2 = \sum\sum\sum (\hat{\tau}\hat{\lambda})_{jk}^2 = SS\ Interaction,$$

$$d.f. = (J - 1)(K - 1)$$

Other useful partitionings include

$$SS\ Main = SS\ Total - SS\ Interaction - SS\ Residual$$

$$d.f. = J + K - 2$$

Note also that, *when all cell frequencies are equal*, i.e. the number of observations in each combination of treatments is the same,

$$SS\ Main = SS\ Columns + SS\ Rows.$$

This will not necessarily be true otherwise. The fact that it is true in a balanced design is one of its main advantages.

Another useful partitioning is

$$SS\ Cells = SS\ Explained = SS\ Main + SS\ Interaction = SS\ Total - SS\ Error$$

$$d.f. = JK - 1$$

When all cell frequencies are equal,

$$SS\ Cells = SS\ Columns + SS\ Rows + SS\ Interaction.$$

Finally, note that,

$$Total\ SS = SS\ Main + SS\ Interactions + SS\ Error = SS\ Explained + SS\ Error$$

$$d.f. = J - 1 + K - 1 + JK + 1 - J - K + N - JK = N - 1$$

Again, *when all cell frequencies are equal*,

$$Total\ SS = SS\ Columns + SS\ Rows + SS\ Interaction + SS\ Error.$$

    E.   *When doing statistical inference, we assume that*
- ✓   for each treatment combination JK, the random error terms ε_{ijk} are ~ $N(0, \sigma^2)$; the variance σ^2 is the same for each treatment combination.
- ✓   the random error terms are independent

II.   TESTS OF INTEREST:

    A. H_0: $(\tau\lambda)_{jk} = 0$ for all j, k
           H_A: $(\tau\lambda)_{jk} <> 0$ for at least 1 j, k

This is a test of whether there are any <u>interaction effects</u>; the appropriate <u>test statistic</u> is

$$F_{(J-1)(K-1),N-JK} = \frac{SS\ Interaction/(J-1)(K-1)}{SS\ Error/(N-JK)} = \frac{MS\ Interaction}{MS\ Error}$$

If the null hypothesis is true, $F \sim \mathbf{F}([J-1][K-1], N-JK)$

    B. H_0: $\tau_1 = \tau_2 = ... = \tau_J = 0$
           H_A: At least 1 $\tau_j <> 0$

This tests whether there are any <u>row effects</u>. The appropriate <u>test statistic</u> is

$$F_{J-1, N-JK} = \frac{SS\ Rows/(J-1)}{SS\ Error/(N-JK)} = \frac{MS\ Rows}{MS\ Error}$$

If the null hypothesis is true, F ~ **F**([J - 1], N - JK)

     C. H_0: $\lambda_1 = \lambda_2 = ... = \lambda_K = 0$
           H_A: At least 1 $\lambda_k <> 0$

This tests whether there are any <u>column effects</u>. The appropriate <u>test statistic</u> is

$$F_{K-1, N-JK} = \frac{SS\ Columns/(K-1)}{SS\ Error/(N-JK)} = \frac{MS\ Columns}{MS\ Error}$$

If the null hypothesis is true, F ~ **F**([K - 1], N - JK).

     NOTE:The last two tests are primarily of interest if you conclude that interaction effects are <u>not</u> significant. If, on the other hand, you conclude that the interaction effects do not equal zero, then you know both treatments (i.e. the row and column effects) are significant.

     D. H_0:   All τ's and λ's $= 0$
           H_A: At least one τ or λ does not equal 0

This tests whether any of the <u>main effects</u> (i.e. row or column effects; or, non-interaction effects) are nonzero. The appropriate <u>test statistic</u> is

$$F_{J+K-2, N-JK} = \frac{SS\ Main/(J+K-2)}{SS\ Error/(N-JK)} = \frac{MS\ Main}{MS\ Error}$$

If the null hypothesis is true, F ~ **F**([J + K - 2], N - JK).

     E. H_0:   All τ's, λ's, and $(\tau\lambda)$'s $= 0$
           H_A: At least one τ, λ, or $(\tau\lambda)$ does not equal 0

     This tests whether there are any effects at all. If the null hypothesis is true, then every cell in the table will have the same true mean. The appropriate <u>test statistic</u> is

$$F_{JK-1, N-JK} = \frac{SS\ Cells/(JK-1)}{SS\ Error/(N-JK)} = \frac{MS\ Cells}{MS\ Error}$$

If the null hypothesis is true, F ~ **F**([JK - 1], N - JK).

III.   ROW, COLUMN, AND INTERACTION EFFECTS – EXAMPLES

What are interaction effects? Here are some substantive examples:

✓ Medicines A and B may have no effect when either is taken alone. But, the two together may have an effect. "The whole is different from the sum of the parts."

✓ Another example: we might find that greater income leads to greater fertility for those who want children, and lower fertility for those who do not want children. We say that the effect of income is dependent on desires, or that desires and income interact in determining fertility.

✓ Good teachers and small classrooms might both encourage learning. A good teacher in a small classroom might be especially effective. The whole is greater than the sum of the parts.

Following are hypothetical 2-way ANOVA examples. The dependent variable is income (in thousands of dollars), the row variable is gender (Male or Female), the column variable is type of occupation (A, B, or C). Unless otherwise stated, assume that frequencies are equal for all cells.

1.   Row (Gender) effects only.

	Occ A	Occ B	Occ C	
Male	$\mu_{MA} = 18$ $\tau\lambda_{MA} = 0$	$\mu_{MB} = 18$ $\tau\lambda_{MB} = 0$	$\mu_{MC} = 18$ $\tau\lambda_{MC} = 0$	$\mu_M = 18$ $\tau_M = 2$
Female	$\mu_{FA} = 14$ $\tau\lambda_{FA} = 0$	$\mu_{FB} = 14$ $\tau\lambda_{FB} = 0$	$\mu_{FC} = 14$ $\tau\lambda_{FC} = 0$	$\mu_F = 14$ $\tau_F = -2$
	$\mu_A = 16$ $\lambda_A = 0$	$\mu_B = 16$ $\lambda_B = 0$	$\mu_C = 16$ $\lambda_C = 0$	$\mu = 16$

The 2 rows differ, but the three columns are all the same. Within each occupation, men make $4,000 more on average than do women; each of the three occupations pays equally well.

2.   Column (Occupation) effects only.

	Occ A	Occ B	Occ C	
Male	$\mu_{MA} = 12$ $\tau\lambda_{MA} = 0$	$\mu_{MB} = 16$ $\tau\lambda_{MB} = 0$	$\mu_{MC} = 20$ $\tau\lambda_{MC} = 0$	$\mu_M = 16$ $\tau_M = 0$
Female	$\mu_{FA} = 12$ $\tau\lambda_{FA} = 0$	$\mu_{FB} = 16$ $\tau\lambda_{FB} = 0$	$\mu_{FC} = 20$ $\tau\lambda_{FC} = 0$	$\mu_F = 16$ $\tau_F = 0$
	$\mu_A = 12$ $\lambda_A = -4$	$\mu_B = 16$ $\lambda_B = 0$	$\mu_C = 20$ $\lambda_C = 4$	$\mu = 16$

The three columns differ, but the two rows are the same. Occupation C pays better than B and B pays better than A. Within each occupation, however, men and women make the same.

3.   Row and column effects.

	Occ A	Occ B	Occ C	
Male	$\mu_{MA} = 14$ $\tau\lambda_{MA} = 0$	$\mu_{MB} = 18$ $\tau\lambda_{MB} = 0$	$\mu_{MC} = 22$ $\tau\lambda_{MC} = 0$	$\mu_{M} = 18$ $\tau_{M} = 2$
Female	$\mu_{FA} = 10$ $\tau\lambda_{FA} = 0$	$\mu_{FB} = 14$ $\tau\lambda_{FB} = 0$	$\mu_{FC} = 18$ $\tau\lambda_{FC} = 0$	$\mu_{F} = 14$ $\tau_{F} = -2$
	$\mu_{A} = 12$ $\lambda_{A} = -4$	$\mu_{B} = 16$ $\lambda_{B} = 0$	$\mu_{C} = 20$ $\lambda_{C} = 4$	$\mu = 16$

Both the rows and columns differ. Within each occupation, men make $4,000 more on average than women do. Within each gender, those in occupation C average $4,000 more than those in B, and those in B average $4,000 more than those in A.

4.   Interaction effects I.

	Occ A	Occ B	Occ C	
Male	$\mu_{MA} = 15$ $\tau\lambda_{MA} = -1$	$\mu_{MB} = 15$ $\tau\lambda_{MB} = -1$	$\mu_{MC} = 21$ $\tau\lambda_{MC} = 2$	$\mu_{M} = 17$ $\tau_{M} = 1$
Female	$\mu_{FA} = 15$ $\tau\lambda_{FA} = 1$	$\mu_{FB} = 15$ $\tau\lambda_{FB} = 1$	$\mu_{FC} = 15$ $\tau\lambda_{FC} = -2$	$\mu_{F} = 15$ $\tau_{F} = -1$
	$\mu_{A} = 15$ $\tau_{A} = -1$	$\mu_{B} = 15$ $\tau_{B} = -1$	$\mu_{C} = 18$ $\tau_{C} = 2$	$\mu = 16$

Five of the six cells have the same mean. However, for some reason, the combination of males and occupation C results in high male earnings.

5.   Interaction effects II - differing magnitudes of effects.

	Occ A	Occ B	Occ C	
Male	$\mu_{MA} = 12$ $\tau\lambda_{MA} = -1$	$\mu_{MB} = 16$ $\tau\lambda_{MB} = -1$	$\mu_{MC} = 26$ $\tau\lambda_{MC} = 2$	$\mu_{M} = 18$ $\tau_{M} = 2$
Female	$\mu_{FA} = 10$ $\tau\lambda_{FA} = 1$	$\mu_{FB} = 14$ $\tau\lambda_{FB} = 1$	$\mu_{FC} = 18$ $\tau\lambda_{FC} = -2$	$\mu_{F} = 14$ $\tau_{F} = -2$
	$\mu_{A} = 11$ $\lambda_{A} = -5$	$\mu_{B} = 15$ $\lambda_{B} = -1$	$\mu_{C} = 22$ $\lambda_{C} = 6$	$\mu = 16$

Men make more than women, and the advantage is especially great in occupation C. Or, those in occupation C make more than those in other occupations, and the advantage is especially great for men.

6.   Interaction effects III - differing directions of effects.

	Occ A	Occ B	Occ C	
Male	$\mu_{MA} = 18$ $\tau\lambda_{MA} = +2$	$\mu_{MB} = 16$ $\tau\lambda_{MB} = 0$	$\mu_{MC} = 14$ $\tau\lambda_{MC} = -2$	$\mu_M = 16$ $\tau_M = 0$
Female	$\mu_{FA} = 14$ $\tau\lambda_{FA} = -2$	$\mu_{FB} = 16$ $\tau\lambda_{FB} = 0$	$\mu_{FC} = 18$ $\tau\lambda_{FC} = 2$	$\mu_F = 16$ $\tau_F = 0$
	$\mu_A = 16$ $\lambda_A = 0$	$\mu_B = 16$ $\lambda_B = 0$	$\mu_C = 16$ $\lambda_C = 0$	$\mu = 16$

In this example, the effect of gender depends on occupation. Males do better than women in Occupation A but worse in occupation C; in Occupation B there is no difference. Or, occupation C is better paying for women but not for men, whereas for occupation A the opposite is true. Note that, if you only looked at the main effects, you would erroneously conclude that gender and occupation have no effects on income, when in reality they do have effects but the effects work in opposing directions.

IV.   Computational Procedures - Two-Way Anova – Balanced Designs

Let A = row variable, B = column variable, J = number of categories for A, K = number of categories for B, T_{Aj} = the sum of the scores in group A_j, T_{Bk} = the sum of the scores in group B_k, T_{AjBk} is the sum of the scores for the observations which fall in both groups A_j and B_k (there are J*K of these totals), n_{Aj} = number of observations in group A_j, n_{Bk} = number of observations in group B_k, n_{AjBk} is the number of observations which fall in both groups A_j and B_k.   [NOTE: While I will show you how to do the raw data calculations, in practice they are tedious enough that I generally would not expect you to do them by hand, at least on an exam. You should know how to do the other formulas, however, as they show how the different parts of the ANOVA table are related to each other.]

Note that many (albeit not all) of the formulas for raw data calculations and Sums of Squares assume a *balanced design*, i.e. all cell frequencies are equal for each possible combination of values for the row and column variables. Computations are somewhat more complicated when designs are not balanced. *The Mean Square formulas and the F tests are accurate regardless of whether the design is balanced or not.*

Formula	Explanation
Raw Data Calculations (Balanced Design)	
(1) = $(\Sigma\Sigma\Sigma y_{ijk})^2/n = N\hat{\mu}^2$	Sum all the observations. Square the result. Divide by the total number of observations.
(2) = $\Sigma\Sigma\Sigma y_{ijk}^2$	Square each observation. Sum the squared observations.
(3) = $\Sigma T_{Aj}^2/n_{Aj}$	Add up the values for the observations for group A_1. Square the result. Divide by the number of observations in group A_1. Repeat for each category of A. Add the results for each of the J groups together.
(4) = $\Sigma T_{Bk}^2/n_{Bk}$	Add up the values for the observations for group B_1. Square the result. Divide by the number of observations in group B_1. Repeat for each category of B. Add the results for each of the K groups together.
(5) = $\Sigma\Sigma T_{AjBk}^2/n_{AjBk}$	Add up the values for the observations which fall in both group A_1 and B_1. Square this value, and divide by n_{A1B1}. Repeat for each of the J*K combinations, and sum the results.

Sums of Squares Calculations (Balanced Design)	
SS Total = (2) - (1)	Total sum of squares
SS Rows = (3) - (1)	Row sum of squares. This is also sometimes called SS_A.
SS Columns = (4) - (1)	Column sum of squares. Also called SS_B.
SS Interaction = (5) + (1) - (3) - (4) = SS Total - SS Rows - SS Columns - SS Error = SS Total – SS Main – SS Error	Interaction sum of squares. Also called SS_{AB}. It may be easier to use the second formula.
SS Error = (2) - (5) = SS Total - SS Cells	Error sum of squares. It is analogous to SS Within in one-way ANOVA. Also called SS Residual.
SS Main = (3) + (4) – [2 * (1)] = SS Columns + SS Rows = SS Total – SS Error – SS Interaction	Main effects Sum of Squares. Also called SS_{A+B}
SS Cells = (5) - (1) = SS Main + SS interaction = SS Total - SS Error.	This is analogous to SS Between in one-way ANOVA. Also called SS Explained.
Mean Square Calculations (Balanced or unbalanced)	
MS Total = s^2 = SS Total/(n-1)	Remember that MS Total = s^2
MS Rows = SS Rows/(J-1)	Also called MS_A.
MS Columns = SS Columns/(K-1)	Also called MS_B.
MS Interaction = SS Interaction/((J-1)(K-1))	Also called MS_{AB}
MS Main = SS Main/(J + K - 2)	Also called MS_{A+B}
MS Cells = SS Cells/((J*K)-1)	Also called MS Explained.
MS Error = SS Error/ (n - J*K)	Also called MS Residual.

Possible F Tests (Balanced or unbalanced):	
MS Rows/MS Error	Do means differ across categories of the row variable, i.e. do tau's differ? d.f. = J-1, n-J*K
MS Columns/MS Error	Do means differ across categories of the column variable, i.e. do lambdas differ? d.f. = K-1, n-J*K
MS Interaction/MS Error	Do any of the interaction effects differ from zero? d.f. = (J-1)(K-1), n-J*K
MS Main/MS Error	Are any of the row or column effects nonzero? d.f. = J + K - 2, n-J*K
MS Cells/MS Error	Are there any differences anywhere across groups? d.f. = (JK-1), N-JK.

An ANOVA table often looks something like this (with the computed values substituted).

Source	SS	D.F.	Mean Square	F
A + B (or Main Effects)	SS Main	J + K - 2	$\frac{\text{SS Main}}{(J + K - 2)}$	$\frac{\text{MS Main}}{\text{MS Error}}$
A (or main effect of A)	SS Rows	J - 1	$\frac{\text{SS Rows}}{(J - 1)}$	$\frac{\text{MS Rows}}{\text{MS Error}}$
B (or main effect of B)	SS Columns	K - 1	$\frac{\text{SS Columns}}{(K - 1)}$	$\frac{\text{MS Columns}}{\text{MS Error}}$
AB (or 2-way interaction)	SS Interaction	(J - 1) * (K - 1)	$\frac{\text{SS Interaction}}{(J -1)(K - 1)}$	$\frac{\text{MS Interaction}}{\text{MS Error}}$
A + B + AB (or explained)	SS Cells	(J * K) - 1	$\frac{\text{SS Cells}}{(J * K) - 1}$	$\frac{\text{MS Cells}}{\text{MS Error}}$
Error (or residual)	SS Error	N - (J * K)	$\frac{\text{SS Error}}{(N - J * K)}$	
Total	SS Total	N - 1	$\frac{\text{SS Total}}{(N - 1)}$	

V.   EXAMPLES.

1.   A researcher is interested in differences in income by Region (North, South, East, and West) and Religion (Catholic, Protestant, Other). She draws a sample of ten people for each combination of region and religion. She finds that SS Rows = 200, SS Columns = 170, SS Interaction = 100, and s^2 = 16.81. Construct the Anova Table, and indicate which effects are significant at the .05 level. (NOTE: Region is the row variable.)

Solution. Again the design is balanced. You don't have to do any work with the raw data here; instead, you have to understand how the different parts of the ANOVA table are related to each other. Let us begin with what we are told:

Source	SS	D.F.	Mean Square	F
A + B (or Main Effects)	SS Main =	J + K - 2 =	$\dfrac{\text{SS Main}}{(J + K - 2)}$ =	$\dfrac{\text{MS Main}}{\text{MS Error}}$ =
A (or main effect of A)	SS Rows = **200**	J - 1 =	$\dfrac{\text{SS Rows}}{(J - 1)}$ =	$\dfrac{\text{MS Rows}}{\text{MS Error}}$ =
B (or main effect of B)	SS Columns = **170**	K - 1 =	$\dfrac{\text{SS Columns}}{(K - 1)}$ =	$\dfrac{\text{MS Columns}}{\text{MS Error}}$ =
AB (or 2-way interaction)	SS Intraction = **100**	(J - 1) * (K - 1) =	$\dfrac{\text{SS Intrction}}{(J -1)(K - 1)}$ =	$\dfrac{\text{MS Intrction}}{\text{MS Error}}$ =
A + B + AB (or explained)	SS Cells =	(J * K) - 1 =	$\dfrac{\text{SS Cells}}{(J * K) - 1}$ =	$\dfrac{\text{MS Cells}}{\text{MS Error}}$ =
Error (or residual)	SS Error =	N - (J * K) =	$\dfrac{\text{SS Error}}{(N - J * K)}$ =	
Total	SS Total =	N - 1 =	$\dfrac{\text{SS Total}}{(N - 1)}$ = **16.81**	

We are also told J = 4 (there are 4 regions), K = 3 (3 religions).

✓   We can deduce that N = J*K*10 = 120.
✓   Recall that s^2 = MS Total, and that MS Total = SS Total/(n-1)
     ==> SS Total = s^2 * (N - 1) = 16.81 * 119 = 2000.
✓   SS Main is obtained by adding SS Rows + SS Columns = 200 + 170 = 370.
✓   SS Cells is obtained by adding up SS Columns + SS Rows + SS Interactions
     = 200 + 170 + 100 = 470.
✓   SS Error is obtained by computing SS Total - SS Cells = 2000 - 470 = 1530.
✓   The remaining quantities in the table are obtained by filling in the appropriate values for the formulas. Hence, we get (* = significant at the .05 level):

Source	SS	D.F.	Mean Square	F
A + B (or Main Effects)	SS Main = 370	J + K - 2 = 5	$\frac{SS\ Main}{(J + K - 2)}$ = 74.00	$\frac{MS\ Main}{MS\ Error}$ = 5.22*
A (or main effect of A)	SS Rows = 200	J - 1 = 3	$\frac{SS\ Rows}{(J - 1)}$ = 66.67	$\frac{MS\ Rows}{MS\ Error}$ = 4.71*
B (or main effect of B)	SS Columns = 170	K - 1 = 2	$\frac{SS\ Columns}{(K - 1)}$ = 85.00	$\frac{MS\ Columns}{MS\ Error}$ = 6.0*
AB (or 2-way interaction)	SS Intraction = 100	(J - 1) * (K - 1) = 6	$\frac{SS\ Intrction}{(J -1)(K - 1)}$ = 16.67	$\frac{MS\ Intrction}{MS\ Error}$ = 1.18
A + B + AB (or explained)	SS Cells = 470	(J * K) - 1 = 11	$\frac{SS\ Cells}{(J * K) - 1}$ = 42.73	$\frac{MS\ Cells}{MS\ Error}$ = 3.02*
Error (or residual)	SS Error = 1530	N - (J * K) = 108	$\frac{SS\ Error}{(N - J * K)}$ = 14.17	
Total	SS Total = 2000	N - 1 = 119	$\frac{SS\ Total}{(N - 1)}$ = 16.81	

Conclusion. Interaction effects are not significant, other effects are.

2.   A consumer research firm wants to compare three brands of radial tires (X, Y, and Z) in terms of tread life over different road surfaces. Random samples of four tires of each brand are selected for each of three surfaces (asphalt, concrete, gravel). A machine that can simulate road conditions for each of the road surfaces is used to find the tread life (in thousands of miles) of each tire. Construct an ANOVA table and conduct F-tests for the presence of nonzero brand effects, road surface effects, and interaction effects.

Surface/ Brand	X	Y	Z
Asphalt	36, 39, 39, 38	42, 40, 39, 42	32, 36, 35, 34
Concrete	38, 40, 41, 40	42, 45, 48, 47	37, 33, 33, 34
Gravel	34, 32, 34, 35	34, 34, 30, 31	36, 35, 35, 33

Solution. I'll show you how to work this by hand (just in case your life ever depends on it) although on an exam I'd be more likely to give you something like problem 1 and/or give you finished results and ask you to interpret them. More critically, I'll show you how to do this in SPSS.

Note that the design is balanced. Let A = Road surface, B = Brand. HINT: It is legitimate to subtract a constant from EVERY observation. This will not affect any of the values in the ANOVA table, and it often makes the calculations simpler. I will subtract 30 from each observation, yielding the following table:

Surface/ Brand	X	T_{AjBk}	Y	T_{AjBk}	Z	T_{AjBk}	T_{Aj}
Asphalt	6 9 9 8	32	12 10 9 12	43	2 6 5 4	17	92
Concrete	8 10 11 10	39	12 15 18 17	62	7 3 3 4	17	118
Gravel	4 2 4 5	15	4 4 0 1	9	6 5 5 3	19	43
T_{Bk}		86		114		53	253

$(1) = (\Sigma\Sigma\Sigma y_{ijk})^2/n = 253^2/36 = 1778.03$
$(2) = \Sigma\Sigma\Sigma y_{ijk}^2 = 6^2 + 9^2 + 12^2 + ... + 3^2 = 2451$
$(3) = \Sigma\, T_{Aj}^2/n_{Aj} = 92^2/12 + 118^2/12 + 43^2/12 = 2019.75$
$(4) = \Sigma\, T_{Bk}^2/n_{Bk} = 86^2/12 + 114^2/12 + 53^2/12 = 1933.42$
$(5) = \Sigma\Sigma\, T_{AjBk}^2/n_{AjBk} = 32^2/4 + 39^2/4 + ... + 19^2/4 = 2370.75$

SS Total = (2) - (1) = 2451 - 1778.03 = 672.97
SS Rows = (3) - (1) = 2019.75 - 1778.03 = 241.72
SS Columns = (4) - (1) = 1933.42 - 1778.03 = 155.39
SS Interaction = (5) + (1) - (3) - (4) =
     2370.75 + 1778.03 - 2019.75 - 1933.42 = 195.61
SS Main = SS Rows + SS Columns = 397.11
SS Cells = (5) - (1) = 592.72
SS Error = (2) - (5) = 80.25

ANOVA TABLE:

SOURCE	SS	D.F.	MEAN SQUARE	F
A + B	397.11	4	99.28	33.43*
A	241.72	2	120.86	40.69*
B	155.39	2	77.70	26.16*
AB	195.61	4	48.90	16.46*
A+B+AB	592.72	8	74.09	24.95*
Error	80.25	27	2.97	
Total	672.97	35	19.23	

* = significant at the .05 level.

NOTE:
- To test for the presence of nonzero road effects, the degrees of freedom = 2,27 and we accept H_0 if $F \leq 3.34$.
- To test for the presence of nonzero brand effects, d.f. = 2,27 and we accept H_0 if $F \leq 3.34$.
- To test for the presence of nonzero interaction effects, d.f. = 4,27 and we accept H_0 if $F \leq 2.72$.
- To test for the presence of any nonzero effects, d.f. = 8, 27 and we accept H_0 if $F \leq 2.21$.

SPSS Solution. In SPSS, the ANOVA command can be used for 2-way ANOVA problems. Alas, you have to enter the syntax directly – you can't do it with the pull-down menus – but it isn't too hard. If you are bound and determined to use the pull-down menus, you can use the UNIANOVA routine – which I personally find a little confusing but I haven't used it very much. To use UNIANOVA, select ANALYZE/ GENERAL LINEAR MODEL/ UNIVARIATE. Here is how you can work the above problem using the ANOVA routine.

```
DATA LIST FREE / Surface Brand Treadlif.
BEGIN DATA.
1 1 36
1 1 39
1 1 39
1 1 38
1 2 42
1 2 40
1 2 39
1 2 42
1 3 32
1 3 36
1 3 35
1 3 34
2 1 38
2 1 40
2 1 41
2 1 40
2 2 42
2 2 45
2 2 48
2 2 47
2 3 37
2 3 33
2 3 33
2 3 34
3 1 34
3 1 32
3 1 34
3 1 35
3 2 34
3 2 34
3 2 30
3 2 31
3 3 36
3 3 35
3 3 35
3 3 33
END DATA.

VARIABLE LABELS SURFACE 'Type of Surface' BRAND 'Brand of tire'
              TREADLIF 'Tread life (1000s of miles)'.
VALUE LABELS SURFACE 1 'Asphalt' 2 'Concrete' 3 'Gravel'/
```

```
              BRAND 1 'X' 2 'Y' 3 'Z'.
ANOVA /VARIABLES TREADLIF BY SURFACE (1,3) BRAND (1,3)/ Method = Experimental.
```

ANOVA

ANOVA[a]

			Experimental Method				
			Sum of Squares	df	Mean Square	F	Sig.
TREADLIF Tread life (1000s of miles)	Main Effects	(Combined)	397.111	4	99.278	33.402	.000
		SURFACE Type of Surface	241.722	2	120.861	40.664	.000
		BRAND Brand of tire	155.389	2	77.694	26.140	.000
	2-Way Interactions	SURFACE Type of Surface * BRAND Brand of tire	195.611	4	48.903	16.453	.000
	Model		592.722	8	74.090	24.928	.000
	Residual		80.250	27	2.972		
	Total		672.972	35	19.228		

a. TREADLIF Tread life (1000s of miles) by SURFACE Type of Surface, BRAND Brand of tire

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VI.   N-WAY ANOVA.

It is also possible to address problems where there are more than 2 treatments, e.g. look at the effect of race, sex and religion on income. Things start to get more complicated, of course, but it can be done. Particularly confusing is the fact that you can have 3-way and higher interactions, and it can be difficult to interpret what these mean.

VII.   ANALYSIS OF COVARIANCE.

Finally, I'll just briefly note that sometimes problems involve "treatments" (or independent variables) that have both nominal and interval-level measurement. For example, we might be interested in the effects of sex, race, and years of education on income. One way to do this is through Analysis of Covariance. In ANCOVA, continuous variables (in this case education) are referred to as <u>covariates</u>. However, such problems can also be addressed via regression techniques, and since that is the more common strategy in Sociology that is where we will focus our attention. But, if you ever find yourself reading a lot of work in psychology or education or related fields, you may come across references to ANCOVA.