Math 60710, Introduction to Algebraic Geometry, Problem Set 1, Fall 2017

INSTRUCTIONS: Do at least 6 of these problems. Due Friday, September 15. In these problems, k denotes an algebraically closed field (in many problems, it would be sufficient to require only that k is a field, or an infinite field). As usual, we let $A = k[x_1, \ldots, x_n]$, and we let $k^\times = k - \{0\}$. We denote by $\mathbf{A}^n$ affine space k^n. For an ideal I, $\mathcal{Z}(I)$ is the vanishing set in $\mathbf{A}^n$ of the ideal I, and for $S \subset \mathbf{A}^n$, $I(S)$ is the ideal of functions vanishing on S.

(1) Let $a = (a_1, \ldots, a_n) \in \mathbf{A}^n$, and consider the ideal $\mathfrak{m}_a := (x_1 - a_1, \ldots, x_n - a_n)$. Prove that $\mathfrak{m}_a$ is a maximal ideal of A.

(2) Let f be an element of A. If $f(a) = 0$ for all a in $\mathbf{A}^n$, then prove that $f = 0$.

(3) (i) Let I be an ideal of a ring A. Prove that its radical $\sqrt{I}$ is an ideal of A.
(ii) Prove that the radical of an ideal I of A is a radical ideal.
(iii) Prove that a prime ideal of A is radical.

(4) Let $k[t]$ be a polynomial ring in one variable, and consider the unique k-algebra homomorphism $\phi : k[x, y] \to k[t]$ such that $\phi(x) = t^2$ and $\phi(y) = t^3$. Prove that $\ker(\phi) = (x^3 - y^2)$.

(5) (i) Show that $xy - 1$ and $y^2 - x^3$ are irreducible in $k[x, y]$.
(ii) Show that the maps $\phi : k \to \mathcal{Z}(y^2 - x^3)$ given by $\phi(a) = (a^2, a^3)$ and $\psi : k^\times \to \mathcal{Z}(xy - 1)$ given by $\psi(a) = (a, a^{-1})$ are bijective.

(6) Suppose the characteristic of k is not 2. Show that $x_1^2 + \cdots + x_m^2$ is irreducible in $k[x_1, \ldots, x_n]$ if and only if $m > 2$ (assume $n \geq m$).

(7) D+F, # 3, 15.3: Let i, j be relatively prime positive integers. Let $R = k[x, y]/(x^i - y^j)$, and let K be the fraction field of the integral domain R. Prove that $K \cong k(t)$ and $k[t]$ is the integral closure of R in K.

(8) Consider the ideal $I = (x^2, xy)$ of $k[x, y]$. Prove that $I = (x, y)^2 \cap (x)$ and $\mathcal{Z}(I) = \mathcal{Z}(x)$ (recall from class that if I and J are ideals of a ring R, then $I \cdot J = \{\sum x_r y_r : x_r \in I, y_r \in J\}$ and $I^2 = I \cdot I$).

(9) (Hartshorne, 1.2): Let $Y \in \mathbf{A}^3$ be the set $Y := \{(t, t^2, t^3) : t \in k\}$. Show that Y is an affine variety of dimension 1, and find generators for the ideal $I(Y)$ of functions vanishing on Y. Show that $A(Y) \cong k[t]$, the polynomial ring in one variable.

(10) (Hartshorne, 1.3): Let $Y = \mathcal{Z}(x^2 - yz, xz - z)$ in $\mathbf{A}^3$. Find the decomposition of Y into irreducible components, and find the vanishing ideal for each component.

(11) (Hartshorne, 1.4): If we identify $\mathbf{A}^2 = \mathbf{A}^1 \times \mathbf{A}^1$ in the natural way, show that the Zariski topology on $\mathbf{A}^2$ is not the product topology on $\mathbf{A}^1 \times \mathbf{A}^1$.

(12) Let Y be a subset of a topological space X and assume Y is irreducible in the subspace topology. Show that $\overline{Y}$ is irreducible in the subspace topology.

(13) Show that a Noetherian topological space is quasi-compact, i.e., every open cover has a finite subcover.