Math 60710, Introduction to Algebraic Geometry, Problem Set 2, Fall 2017

INSTRUCTIONS: Do at least 6 of these problems. Due Monday, Oct. 9. Note that I have assigned many but not all problems from Hartshorne. If you want to do a problem from sections two and three of Hartshorne that I have not assigned, you can substitute that problem for one of the problems I have assigned. In these problems, k denotes an algebraically closed field (in many problems, it would be sufficient to require only that k is a field, or an infinite field). As usual, we let $A = k[x_1, \dots, x_n]$, and we let $k^\times = k - \{0\}$. We denote by $\mathbf{A}^n$ affine space k^n. For an ideal I, $\mathcal{Z}(I)$ is the vanishing set in $\mathbf{A}^n$ of the ideal I, and for $S \subset \mathbf{A}^n$, $I(S)$ is the ideal of functions vanishing on S.

(1) Let $\phi : X \to Y$ be a morphism of varieties. Recall that we defined $\phi^* : O(Y) \to O(X)$ by $\phi^*(f) = f \circ \phi$.
(i) Assume that $\phi(X)$ is dense in Y. Show that ϕ^* is injective.
(ii) Let $P \in X$. Show that the map $\phi^* : O_{\phi(P)} \to O_P$ given by $\phi^*(\langle U, f\rangle) = \langle \phi^{-1}(U), \phi^*(f)\rangle$ is a well-defined ring homomorphism.
(iii) Assume that $\phi(X)$ is dense in Y. Define $\phi^* : K(Y) \to K(X)$ using the same formula as in part (i). Show that ϕ^* is a well-defined ring homomorphism.

(2) Let X be a variety, let $W \subset X$ be a nonempty open set, and let $j : W \to X$ be the inclusion.
(i) Let $P \in W$. Show that $j^* : O_P(X) \to O_P(W)$ is an isomorphism, where j^* is defined in part (ii) of the previous problem.
(ii) Show that $j^* : K(X) \to K(W)$ is an isomorphism.

(3) Let $\phi : X \to Y$ be a continuous map of topological spaces. If $V \subset X$ is irreducible, then prove that $\phi(V)$ is irreducible in Y.

(4) Let X, Y be varieties and let $\phi : X \to Y$ be a map of sets. Suppose $X = \cup X_i$ is a union of open sets X_i and $Y = \cup Y_j$ is a union of open sets. Suppose that for each X_i there is $Y_{j(i)}$ such that $\phi(X_i) \subset Y_{j(i)}$, and denote by $\phi_i = \phi|_{X_i} : X_i \to Y_{j(i)}$. Show that if ϕ_i is a morphism for each i, then ϕ is a morphism.

(5) Let $k^\times = \{x \in \mathbf{A}^1 : x \neq 0\}$. Define $\phi : k^\times \to \mathcal{Z}(xy - 1) \subset \mathbf{A}^2$ by $\phi(a) = (a, a^{-1})$. Prove that ϕ is an isomorphism.

(6) Let R be a ring with prime ideal $\mathfrak{p}$, and consider the local ring $R_\mathfrak{p}$. For an ideal I of R, we let $IR_\mathfrak{p} = \{\frac{x}{s} : x \in I, s \in R - \mathfrak{p}\}$. Let $I_{\subset\mathfrak{p}}$ be the collection of prime ideals $\mathfrak{q}$ of R such that $\mathfrak{q} \subset \mathfrak{p}$. Show that for $\mathfrak{q} \in I_{\subset\mathfrak{p}}$, the ideal $\mathfrak{q}R_\mathfrak{p}$ is a prime ideal of $R_\mathfrak{p}$, and that the resulting map from $I_{\subset\mathfrak{p}}$ to prime ideals of $R_\mathfrak{p}$ given by $\mathfrak{q} \mapsto \mathfrak{q}R_\mathfrak{p}$ is bijective.

(7) Exercise 2.5 in Hartshorne (hint: show that if a topological space X is a finite union of Noetherian open subspaces, then X is Noetherian).

(8) Exercise 2.6 in Hartshorne.

(9) Exercise 2.9 in Hartshorne.

(10) Exercise 2.12 in Hartshorne.

(11) Exercise 2.14 in Hartshorne.

(12) Exercise 2.16 in Hartshorne.

(13) Exercise 3.1 in Hartshorne (a conic in $\mathbf{A}^n$ or $\mathbb{P}^n$ is an algebraic set defined by a single homogeneous quadratic polynomial).

(14) Hartshorne, Exercise 3.2.

(15) Hartshorne, Exercise 3.5.

(16) Hartshorne, Exercise 3.6.

(17) Hartshorne, Exercise 3.7.

(18) Hartshorne, Exercise 3.9.
(19) Hartshorne, Exercise 3.10.
(20) Hartshorne, Exericse 3.13.
(21) Hartshorne, Exercise 3.14.
(22) Hartshorne, Exercise 3.17.
(23) Hartshorne, Exercise 3.21.