Math 60710, Introduction to Algebraic Geometry, Problem Set 3, Fall 2017

INSTRUCTIONS: Do at least 6 of these problems. Due Wednesday, Dec. 6. Note that I have assigned many but not all problems from Hartshorne. If you want to do a problem from sections four through seven of Hartshorne that I have not assigned, you can substitute that problem for one of the problems I have assigned. In these problems, k denotes an algebraically closed field. As usual, we let $A = k[x_1, \dots, x_n]$, and we let $k^\times = k - \{0\}$. We denote by $\mathbf{A}^n$ affine space k^n. For an ideal I, $\mathcal{Z}(I)$ is the vanishing set in $\mathbf{A}^n$ of the ideal I, and for $S \subset \mathbf{A}^n$, $I(S)$ is the ideal of functions vanishing on S.

(1) (i) Let $S = k[x, y]$ and let $M = k[x, y]/(x^2, xy)$. Find a filtration of M by graded S-submodules, $M^0 = 0 \subset M^1 \subset M^2 \subset M^3 = M$ such that for $i = 1, 2, 3$, $M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(l_i)$ for a prime ideal $\mathfrak{p}_i$ and an integer l_i (hint: the prime ideals that appear as above are the prime ideals that appear when we write (x^2, xy) as an intersection of powers of prime ideals).
(ii) Let $S = k[x, y]$ and let $M = k[x, y]/(x^3y^2)$. Find a filtration of M as in (i) (you will need more M^i).

(2) Let $S = k[x, y, z]$. For each of the following graded S-modules M, find a filtration of M by graded S-submodules, $M^0 = 0 \subset M^1 \subset M^2 \cdots \subset M^r = M$ such that for $i = 1, 2, r$, $M^i/M^{i-1} \cong (S/\mathfrak{p}_i)(l_i)$ for a prime ideal $\mathfrak{p}_i$ and an integer l_i (hint: a good source for $\mathfrak{p}_i$ comes from looking at $\mathcal{Z}(\mathrm{Ann}(M))$).
(i) $M = S/(y^2z - x^3, x)$ (hint: write the ideal $(y^2z - x^3, x)$ in a different form).
(ii) $M = S/(y^2z - x^3, y)$.
(iii) $M = S/(y^2z - x^3, y - x)$.

(3) Compute the vector fields on $\mathcal{Z}(y^2 - x^3)$. Show that they are all of the form $\xi_{a,b} = a\partial_x + b\partial_y$ where $a, b \in k[x, y]/(y^2 - x^3)$. What conditions must a and b satisfy for $\xi_{a,b}$ to be a vector field. Compute the vector fields on $\mathcal{Z}(y - x^3)$. Show that they are the same as vector fields on $\mathbf{A}^1$.

(4) Let $\phi : \mathbf{A}^n \to \mathbf{A}^m$ be a morphism, and let $\phi = (\phi_1, \dots, \phi_m)$, where $\phi_1, \dots, \phi_m \in A(\mathbf{A}^m)$. Let $x_1, \dots, x_n$ be coordinates on $\mathbf{A}^n$ and let $y_1, \dots, y_m$ be coordinates on $\mathbf{A}^m$.
(i) Show that for each $p \in \mathbf{A}^n$, $d\phi_p^*(dy_j) = \sum_{i=1,\dots,n} \partial_{x_i}(\phi_j)(p)dx_i$.
(ii) Show that for each $p \in \mathbf{A}^n$, $d\phi_p(\partial_{x_i}) = \sum_{j=1}^m \partial_{x_i}(\phi_j)(p)\partial_{y_j}$.
(iii) Consider the determinant map $\phi : M(n, k) \to \mathbf{A}^1$ given by taking $\phi(C) = \mathrm{Det}(C)$. We identify $\mathbf{A}^{n^2} \cong M(n, k)$ by taking the matrix entries as coordinates, and thus for $C \in M(n, k)$, we identify $T_C(M(n, k)) \cong M(n, k)$. Show that for a matrix T, $d\phi_C(T) = \mathrm{trace}(T)$.

(5) Assume that the characteristic of k is not 2.
(i) Let $Q = x_1^2 + \cdots + x_r^2 \in A(\mathbf{A}^n)$, where $r \leq n$. Determine the singular locus of $\mathcal{Z}(Q)$.
(ii) Let $Q = \sum_{i=0}^n x_i^2 \in S = k[x_0, \dots, x_n]$. Find the singular locus of $\mathcal{Z}(Q)$ in ¶^{n}.

(6) (i) Let $Y = \mathcal{Z}(y^2 - x^3) \subset \mathbf{A}^2$. Determine the singular locus of $\overline{Y}$, the closure of Y in ¶^{2}.
(ii) Let $Y = \mathcal{Z}(y^3 - x^5) \subset \mathbf{A}^2$. Determine the singular locus of the closure of Y in ¶^{2}.

(7) Let k be a field of characteristic p, where p is prime. Let $F : \mathbf{A}^1 \to \mathbf{A}^1$ be given by $F(a) = a^p$. For each $a \in \mathbf{A}^1$, compute $dF_a : T_a(\mathbf{A}^1) \to T_{F(a)}(\mathbf{A}^1)$.

(8) Hartshorne, Exercise 4.1.

(9) Hartshorne, Exercise 4.3.

(10) Hartshorne, Exercise 4.4.

(11) Hartshorne, Exercise 4.5.

(12) Hartshorne, Exercise 4.6.

(13) Hartshorne, Exercise 4.7.
(14) Hartshorne, Exercise 4.10 (try this also for a few more curves with a singularity at $(0, 0)$).
(15) Hartshorne, Exercise 5.1.
(16) Hartshorne, Exercise 5.2.
(17) Hartshorne, Exercise 5.3.
(18) Hartshorne, Exercise 5.4.
(19) Hartshorne, Exercise 5.6.