NOTES ON FIBER DIMENSION

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Let $\phi : X \to Y$ be a morphism of affine algebraic sets, defined over an algebraically closed field k. For $y \in Y$, the set $\phi^{-1}(y)$ is called the fiber over y. In these notes, I explain some basic results about the dimension of the fiber over y. These notes are largely taken from Chapters 3 and 4 of Humphreys, "Linear Algebraic Groups", chapter 6 of Bump, "Algebraic Geometry", and Tauvel and Yu, "Lie algebras and algebraic groups". The book by Bump has an incomplete proof of the main fact we are proving (which repeats an incomplete proof from Mumford's notes "The Red Book on varieties and Schemes"). Tauvel and Yu use a step I was not able to verify. The important thing is that you understand the statements and are able to use the Theorems 0.22 and 0.24.

Let A be a ring. If $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_k$ is a chain of distinct prime ideals of A, we say the chain has length k and ends at $\mathfrak{p}$.

Definition 0.1. *Let $\mathfrak{p}$ be a prime ideal of A. We say $ht(\mathfrak{p}) = k$ if there is a chain of distinct prime ideals $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_k = \mathfrak{p}$ in A of length k, and there is no chain of prime ideals in A of length $k+1$ ending at $\mathfrak{p}$.*

If B is a finitely generated integral k-algebra, we set $\dim(B) = \dim(F)$, where F is the fraction field of B.

Theorem 0.2. *(Serre, "Local Algebra", Proposition 15, p. 45) Let A be a finitely generated integral k-algebra and let $\mathfrak{p} \subset A$ be a prime ideal. Then $ht(\mathfrak{p}) = \dim(A) - \dim(A/\mathfrak{p})$.*

Theorem 0.3. *(see Matsumura, "Commutative Ring Theory", Theorem 13.5, p. 100) (i) Let A be a Noetherian ring and let $f \in A$ be a nonzero nonunit. Then if $\mathfrak{p}$ is minimal among prime ideals of A containing f, $ht(\mathfrak{p}) \leq 1$.*

(ii) If A is also an integral domain and $\mathfrak{p}$ is as in (i), then $ht(\mathfrak{p}) = 1$.

To prove (ii) using (i), note that $(0) \subset \mathfrak{p}$ is chain of length 1, so $ht(\mathfrak{p}) \geq 1$.

The above results in commutative algebra are not easy consequences of definitions. You can find approaches to algebraic geometry which get around using them (notably the treatment in Springer's book on Algebraic Groups). My feeling is that it is better to understand these results. For the purposes of this course, you may simply believe them, but if you are going to do mathematics involving algebraic geometry, it is useful to understand these results in commutative algebra.

Theorem 0.4. *("Going Down" theorem)(Matsumura, as above, Theorem 9.4, p. 68, or Dummit and Foote, Theorem 21, p. 668) Let B be an integrally domain and suppose that*

B is integral over a subring A. Suppose that A is integrally closed. Let $\mathfrak{p}_2 \subset \mathfrak{p}_1$ be prime ideals of A and suppose that $\mathfrak{q}_1$ is a prime ideal of B such that $\mathfrak{q}_1 \cap A = \mathfrak{p}_1$. Then there exists a prime ideal $\mathfrak{q}_2$ of B such that $\mathfrak{q}_2 \subset \mathfrak{q}_1$ and $\mathfrak{q}_2 \cap A = \mathfrak{p}_2$.

Theorem 0.5. *(see Eisenbud, "Commutative Algebras with a View Toward Algebraic Geometry", Corollary 13.13). Let A be a finitely generated integral k-algebra with fraction field K. Let B be the integral closure of A. Then B is a finitely generated A-module. In particular, B is a finitely generated integral k-algebra.*

The first assertion of the Theorem is that there exist $b_1, \ldots, b_k \in B$ such that $B = Ab_1 + \cdots + Ab_k$. It follows easily that B is generated as an algebra by $\{b_1, \ldots, b_k\}$ and any set of generators of A.

Lemma 0.6. *(Serre, "Local Algebra", Lemma 1, p. 8) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals of a ring A. If I is an ideal of A and $I \subset \cup_{i=1}^{n} \mathfrak{p}_i$, then $I \subset \mathfrak{p}_i$ for some i.*

Proposition 0.7. *(Tauvel and Yu, Theorem 3.28) (i) Let A be an integrally closed domain. Then $A[t]$ is an integrally closed domain.*

(ii) If A is an integrally closed domain, then $A[t_1, \ldots, t_n]$ is an integrally closed domain.

Proof : Let K be the fraction field of A and let $p(t), q(t)$ be monic polynomials such that $p(t)q(t) \in A[t]$. Then $p(t), q(t) \in A[t]$. Indeed, let M be the algebraic closure of K. Factor $p(t)$ into linear factors in $M[t]$ with roots $\alpha_1, \ldots, \alpha_k$ and factor $q(t)$ into linear factors with roots $\beta_1, \ldots, \beta_r$. If γ is one of the α_i or β_j, then $p \cdot q(\gamma) = 0$. Since $p \cdot q$ is monic in $A[t]$, it follows that γ is integral over A in M. The coefficients of $p(t)$ and $q(t)$ are polynomials in the α_i and β_j and hence are integral over A, and in K. Thus, these coefficients are in A, so $p(t), q(t) \in A[t]$.

Let K and L be the fraction fields of A and $A[t]$ respectively. Let $\alpha \in L$ be integral over $A[t]$. Then α is also integral over $K[t] \supset A[t]$, so since $K[t]$ is a UFD and hence integrally closed, $\alpha \in K[t]$.

Let $Q(x) \in A[t][x]$ be a monic polynomial with $Q(\alpha) = 0$ and write $Q(x) = x^m + p_{m-1}x^{m-1} + \cdots + p_0$, with $p_i \in A[t]$. Choose s so s is larger than the degree of $\alpha = \alpha(t)$ and also so for all $i = 0, \ldots, m-1$, s is larger than the degree of $p_i = p_i(t)$. Let $\beta = \alpha - t^s \in K[t]$, so by the choice of s, $-\beta$ is monic in $K[t]$.

Let $R(x) = Q(x + t^s)$. Then $R(\beta) = Q(\beta + t^s) = Q(\alpha) = 0$. Expand $R(x) = x^m + q_{m-1}x^{m-1} + \cdots + q_0$ as a polynomial in x with coefficients in $A[t]$. The term $q_0 = t^{sm} + p_{m-1}t^{s(m-1)} + \cdots + p_0$ is monic in t by the choice of s. Since $R(\beta) = 0$,

$$q_0 = -\beta(\beta^{m-1} + q_{m-1}\beta^{m-2} + \cdots + q_1) = -\beta \cdot \gamma, \gamma = (\beta^{m-1} + q_{m-1}\beta^{m-2} + \cdots + q_1) \in K[t].$$

Since q_0 and $-\beta$ are monic in $K[t]$, it follows that γ is monic in $K[t]$. Since $q_0 \in A[t]$, it follows from the assertion of the first paragraph of this proof that $-\beta \in A[t]$. Hence, $\alpha = \beta + t^s \in A[t]$, which establishes the first assertion. The second assertion follows by an easy induction.

Q.E.D.

Let X be an affine variety and let $f \in k[X]$ be a nonzero nonunit. Then the zero set $V(f)$ is called a hypersurface.

If X is an affine variety and $Y \subset X$ is a closed subvariety, then $\operatorname{co}\dim_X(Y) = \dim(X) - \dim(Y)$ is the codimension of Y in X. It is clear that if $Z \subset Y$ is a closed subvariety, then $\operatorname{co}\dim_X(Z) = \operatorname{co}\dim_X(Y) + \operatorname{co}\dim_Y(Z)$.

Theorem 0.8. *Let X be an affine variety and let $Z \subset X$ be a closed subvariety. Then $\operatorname{co}\dim_X(Z) = 1$ if and only if Z is an irreducible component of a hypersurface $V(f) \subset X$.*

Proof: Suppose $\operatorname{co}\dim_X(Z) = 1$. Since $Z \neq X$, $I(Z)$ is a nonzero prime ideal of $k[X]$, so we can choose nonzero $f \in I(Z)$. Clearly, f is not a unit. Thus, $V(f)$ is a proper subset of X and $Z \subset V(f)$. Let $V(f) = Y_1 \cup \cdots \cup Y_k$ be the irreducible components of the hypersurface $V(f)$. Then $Z \subset Y_i$ for some i. But $\dim(Y_i) < \dim(X)$, so the hypothesis implies that $\dim(Y_i) \leq \dim(Z)$. It follows that $Y_i = Z$.

Now suppose that Z is an irreducible component of a hypersurface $V(f)$. Thus, $f \in I(Z)$, and $I(Z)$ is prime. If $f \in J \subset I(Z)$, where J is prime, then $Z \subset V(J) \subset V(f)$, so since $V(J)$ is irreducible, $Z = V(J)$. Hence, $J = I(Z)$. Thus, $I(Z)$ is minimal among all prime ideals of $k[X]$ containing f. Hence by Theorem 0.3, $ht(I(Z)) = 1$, so $\dim(Z) = \dim(k[Z]) = \dim(k[X]/I(Z)) = \dim(k[X]) - 1 = \dim(X) - 1$ by Thereom 0.2.

Q.E.D.

This last theorem can be proved without using 0.2 and 0.3 if $X = k^n$ since then we can reduce to the case when $V(f)$ is irreducible.

Corollary 0.9. *Let X be an affine variety and let Y be a closed subvariety. If $\operatorname{co}\dim_X(Y) = r$, then for each $i = 1, \ldots, r$, there exists a closed subvariety Y_i of X such that $\operatorname{co}\dim_X(Y_i) = i$ and $Y = Y_r \subset Y_{r-1} \subset \cdots \subset Y_1$.*

Proof: This is clear in case $r = 1$, and we argue by induction on r. Since $Y \neq X$, there exists nonzero $f \in I(Y)$. Since f is a nonunit, by Theorem 0.8, all irreducible components of $V(f)$ have codimension 1 in X. Since Y is irreducible and $Y \subset V(f)$, it follows that $Y \subset Z_0$ for some irreducible component Z_0 of $V(f)$. Take $Y_1 = Z_0$. Then $\operatorname{co}\dim_{Y_1}(Y) = r - 1$, so by induction, there exists a chain of closed subvarieties $Y_r = Y \subset Y_{r-1} \subset \cdots \subset Y_2$ in Y_1 such that $\operatorname{co}\dim_{Y_1}(Y_i) = i - 1$. Then $Y_r \subset \cdots \subset Y_1$ is the desired chain in X.

Q.E.D.

Corollary 0.10. *Let X be an affine variety and let $f_1, \ldots, f_r \in k[X]$. If Z is an irreducible component of $V(f_1, \ldots, f_r)$, then $\operatorname{co}\dim_X(Z) \leq r$.*

Proof: Proceed by induction on r, noting that the case $r = 1$ follows by Theorem 0.8. Let Z be an irreducible component of $V(f_1, \ldots, f_r)$. Since $Z \subset V(f_1)$, $Z \subset Y$ for some irreducible component Y of $V(f_1)$. Let $\overline{f_2}, \ldots, \overline{f_r}$ be the images of $f_2, \ldots, f_r$ in $k[Y]$. Then $Z \subset V(\overline{f_2}, \ldots, \overline{f_r}) \subset V(f_1, \ldots, f_r)$. Since Z is a maximal closed irreducible subset of $V(f_1, \dots, f_r)$, it follows that Z is an irreducible component of $V(\overline{f_2}, \dots, \overline{f_r})$ contained in Y. Apply the inductive assumption with Y in place of X to obtain $\operatorname{co dim}_Y(Z) \leq r-1$. Since $\operatorname{co dim}_X(Y) \leq 1$ by Theorem 0.8, we get $\operatorname{co dim}_X(Z) \leq r$.

Q.E.D.

Corollary 0.11. *Let X be an affine variety and let $Y \subset X$ be a closed subvariety. Suppose $\operatorname{co dim}_X(Y) = r$. Then Y is an irreducible component of $V(f_1, \dots, f_r)$ for some $f_1, \dots, f_r \in k[X]$.*

Proof: We prove the following stronger statement:

(*) if we are given a chain of closed subvarieties $Y_r \subset \cdots \subset Y_1$ with $\operatorname{co dim}_X(Y_i) = i$, then there exists $f_1, \dots, f_r \in k[X]$ such that for all $1 \leq q \leq r$, every irreducible component of $V(f_1, \dots, f_q)$ has codimension q in X and Y_q is an irreducible component of $V(f_1, \dots, f_q)$. If we assume (*) and take $Y_r = Y$ and use 0.9 to find a chain $Y_r \subset \cdots \subset Y_1$ as above, we complete the proof.

Prove (*) by induction on q. If $q = 1$, then by Theorem 0.8, Y_1 is an irreducible component of $V(f)$ for some nonzero nonunit $f \in k[X]$, and every irreducible component of $V(f)$ has codimension one in X. Now assume we have found $f_1, \dots, f_{q-1}$ as above. Let $Z_1 = Y_{q-1}, Z_2, \dots, Z_m$ be the irreducible components of $V(f_1, \dots, f_{q-1})$, so $\operatorname{co dim}_X(Z_j) = q-1$ for all j. Hence, $\dim(Y_q) < \dim(Z_j)$, so no $Z_j \subset Y_q$, so $I(Y_q) \not\subset I(Z_j)$ for all j. Thus, $I(Y_q) \not\subset I(Z_1) \cup \cdots \cup I(Z_m)$ by Lemma 0.6. Hence, there exists $f_q \in I(Y_q)$ such that $f_q \notin I(Z_j)$ for all j. Thus, $\overline{f_q}$ is a nonzero nonunit in $k[Z_j]$ for all j, where $\overline{f_q}$ is the image of f_q in $k[Z_j]$. Hence, by Theorem 0.8 applied to $\overline{f_q} \in k[Z_j]$, every irreducible component W of $V(\overline{f_q}) = V(f_q) \cap Z_j$ has $\operatorname{co dim}_{Z_j}(W) = 1$. Thus, $\operatorname{co dim}_X(W) = q$. Let V be an irreducible component of $V(f_1, \dots, f_q)$. Then $V \subset V(f_1, \dots, f_{q-1})$, and it follows that V is in some irreducible component Z_j. Thus, $V \subset V(f_q) \cap Z_j$. Hence $V \subset W$, where W is an irreducible component of $V(f_q) \cap Z_j$. But $\operatorname{co dim}_X(V) \leq q$ by Corollary 0.10, so $\dim(V) \geq \dim(W)$, and hence $V = W$. Finally, note that $Y_q \subset V(f_1, \dots, f_q)$, so since Y_q is irreducible, Y_q is a subset of one of the above irreducible components V. Since $\dim(Y_q) = \dim(V)$, $Y_q = V$.

Q.E.D.

Theorem 0.12. *Let $\phi : X \to Y$ be a dominant morphism of affine varieties. Let $r = \dim(X) - \dim(Y)$. Let $W \subset Y$ be a closed subvariety and let Z be an irreducible component of $\phi^{-1}(W)$. If $\overline{\phi(Z)} = W$, then $\dim(Z) \geq \dim(W) + r$. In particular, if $y \in \phi(X)$, then each irreducible component of the fiber $\phi^{-1}(y)$ has dimension at least r.*

Proof: Let $s = \operatorname{co dim}_Y(W)$. By Corollary 0.11, W is an irreducible component of $V(f_1, \dots, f_s)$ for some $f_1, \dots, f_s \in k[Y]$. Let $g_i = \phi^*(f_i)$ for $i = 1, \dots, s$. It follows from definitions that $Z \subset V(g_1, \dots, g_s)$. Since Z is irreducible, $Z \subset Z_0$, where Z_0 is an irreducible component of $V(g_1, \dots, g_s)$. Again from definitions, $\phi(V(g_1, \dots, g_s)) \subset V(f_1, \dots, f_s)$, so $\overline{\phi(Z)} \subset \overline{\phi(Z_0)} \subset V(f_1, \dots, f_s)$. Note that $\overline{\phi(Z_0)}$ is irreducible by Exercise

8, problem set 1. Since $W = \overline{\phi(Z)}$ by assumption, $W \subset \overline{\phi(Z_0)}$, so $W = \overline{\phi(Z_0)}$ by definition of irreducible component. Thus, $\phi(Z_0) \subset W$, so $Z_0 \subset \phi^{-1}(W)$.

But Z is an irreducible component of $\phi^{-1}(W)$, so $Z = Z_0$. Hence Z is an irreducible component of $V(g_1, \dots, g_s)$ so by Corollary 0.10, $\operatorname{co}\dim_X(Z) \leq s$. Thus,
$\dim(Z) \geq \dim(X) - s = r + \dim(Y) - s = r + \dim(W)$.

Q.E.D.

Remark 0.13. *An affine variety X is called normal if $k[X]$ is integrally closed in $k(X)$. The affine variety k^n is normal since a unique factorization domain is integrally closed.*

Remark 0.14. *Let X be a normal affine variety. Then $X \times k^n$ is a normal variety. Indeed, $k[X \times k^n] = k[X][t_1, \dots, t_n]$ is integrally closed by Proposition 0.7.*

Proposition 0.15. *Let $\phi : X \to Y$ be a finite dominant morphism of affine varieties. Then*

(i) ϕ is surjective.

(ii) If $y \in Y$, then the fiber $\phi^{-1}(y)$ is finite.

(iii) Let $Z \subset X$ be a closed subvariety. Then $\phi(Z)$ is closed, $\dim(Z) = \dim(\phi(Z))$*, and $\phi : Z \to \phi(Z)$ is finite.*

(iv) Suppose that Y is normal and let $W \subset Y$ be a closed subvariety. For any irreducible component Z of $\phi^{-1}(W)$, $\phi(Z) = W$ and $\dim(Z) = \dim(W)$.

Proof: We leave (i) as an exercise. For (iii), let $\psi = \phi|_Z : Z \to V := \overline{\phi(Z)}$ be the restriction of ϕ to Z. We claim that ψ is finite. Given the claim, since ψ is clearly dominant, ψ is surjective, so $\phi(Z) = \psi(Z) = V$, so $\phi(Z)$ is closed. Thus, $k[Z]$ is integral over $k[V]$ so $k(Z)$ is algebraic over $k(V)$, and it follows from properties of transcendence bases that $\dim(Z) = \dim(V)$. Thus, we have reduced (iii) to the claim.

For the claim, let $i : Z \to X$ and let $j : V \to Y$ be the inclusions. Then since $\phi \circ i = j \circ \psi$, $\psi^* j^* = i^* \phi^*$. Thus, if $f \in k[Y]$,
(*)$\phi^*(f) + I(Z) = \psi^*(f + I(V))$ in $k[Z] = k[X]/I(Z)$ (this uses $i^* : k[X] \to k[Z]$ takes the quotient by $I(Z)$ and corresponding result for j^*). Since ϕ is finite, $k[X]$ is a finitely generated $\phi^* k[Y]$-module. Thus, there exist $a_1, \dots, a_k \in k[X]$ such that for each $f \in k[X]$, there exist $f_1, \dots, f_q \in k[Y]$ such that
(**) $f = \sum_{i=1}^k \phi^*(f_i) a_i$.

This identity is still true in $k[X]/I(Z)$, so using (*) above,
$f = \sum_{i=1}^k \psi^*(f_i + I(V)) \cdot a_i + I(Z)$ in $k[Z]$. Thus, $k[Z]$ is a finitely generated $\psi^* k[V]$ module, so ψ is finite. This completes the proof of (iii).

For (iv), let $W \subset Y$ be a closed subvariety and let $Z \subset \phi^{-1}(W)$ be an irreducible component. We claim that $\dim(Z) \geq \dim(W)$. Thus, by (iii), $\dim(Z) = \dim(\phi(Z)) \geq$

$\dim(W)$. Since $\phi(Z)$ is closed and $\phi(Z) \subset W$, it follows that $\phi(Z) = W$, and hence $\dim(Z) = \dim(W)$. This reduces (iv) to the claim.

To verify the claim, recall that if $V \subset X$ is a closed subvariety, then it follows from definitions that

(*) ${\phi^*}^{-1} I(V) = I(\phi(V))$. In our situation, ϕ^* is injective since ϕ is dominant, so $I(V) \cap \phi^* k[Y] = \phi^*(I(\phi(V)))$.

Let $B = k[X]$ and let $A = \phi^*(k[Y])$. Let $I = I(Z)$ and let $J = \phi^*(I(W))$. Since $\phi(Z) \subset W$, $I(W) \subset I(\phi(Z))$, so by (*), $I(Z) \cap A = \phi^*(I(\phi(Z))) \supset J$.

We apply the Going Down Theorem to the extension $B \supset A$. Let $\alpha \in X$ be a point of $\phi^{-1}(W)$ contained in Z but in no other irreducible component of $\phi^{-1}(W)$ and let $\mathfrak{m}_\alpha \subset B$ and $\mathfrak{m}_{\phi(\alpha)} \subset k[Y]$ be the corresponding maximal ideals. By (*), $\mathfrak{m}_\alpha \cap A = \phi^*(\mathfrak{m}_{\phi(\alpha)})$. Since $\phi(\alpha) \in W$, $I(W) \subset \mathfrak{m}_{\phi(\alpha)}$, so $J \subset \phi^*(\mathfrak{m}_{\phi(\alpha)})$ in A. By the Going Down Theorem, there exists a prime ideal $\mathfrak{p}_1$ of B such that $\mathfrak{p}_1 \subset \mathfrak{m}_\alpha$ and $\mathfrak{p}_1 \cap A = J$. Then $\alpha \in V(\mathfrak{p}_1)$, so since $V(\mathfrak{p}_1)$ is irreducible, by the choice of α, $V(\mathfrak{p}_1) \subset Z$. Thus, $I(Z) \subset \mathfrak{p}_1$, so $I \cap A \subset \mathfrak{p}_1 \cap A = J$. Hence, $I \cap A = J$. and the induced morphism $A/J \to B/I$ is injective. Hence, $\phi^* : k[W] \to k[Z]$ is injective, and it follows from properties of transcendence degree that $\dim(W) \leq \dim(Z)$, verifying the claim.

For (ii), let Z be an irreducible component of $\phi^{-1}(y)$. By (iii), $\dim(Z) = \dim(\phi(Z)) = \dim(y) = 0$. Therefore, Z is a point, which implies (ii).

Q.E.D.

To proceed, we need to establish some results about normal varieties (see Tauvel and Yu, Chapter 17).

Let $\phi : X \to Y$ be a dominant morphism of affine varieties. Then $\phi^* : k[Y] \to k[X]$ is injective, so there is an induced field homomorphism $\phi^* : k(Y) \to k(X)$.

Remark 0.16. *A dominant morphism $\phi : X \to Y$ is called birational if $\phi^* : k(Y) \to k(X)$ is an isomorphism of fields.*

For example, the morphism $\phi : k \to V(y^2 - x^3)$ given by $\phi(a) = (a^2, a^3)$ is birational, but is not an isomorphism.

Proposition 0.17. *Let $\phi : X \to Y$ be a birational morphism of affine varieties. Then there exists nonzero $s \in k[Y]$ such that $\phi : X_{\phi^*(s)} \to Y_s$ is an isomorphism of affine varieties. In particular, a dominant morphism $\phi : X \to Y$ of affine varieties is birational if and only if there is a nonempty open set $U \subset Y$ such that $\phi : \phi^{-1}(U) \to U$ is an isomorphism.*

Proof : Let $k[X] = k[\alpha_1, \ldots, \alpha_m]$. Since $k[X] \subset k(X)$, by hypothesis, there exist $b_1, \ldots, b_m \in k(Y)$ such that $\phi^*(b_i) = \alpha_i$. Then each $b_i = \frac{\gamma_i}{s_i}$ for some $\gamma_i, s_i \neq 0 \in k[Y]$. Write $b_i = \frac{\beta_i}{s}$ with $s = s_1 \cdots s_m$, and $\beta_i = \frac{s\gamma_i}{s_i} \in k[Y]$. Then $\phi^* : k[Y]_s \to k[X]_{\phi^*(s)}$ is easily seen to be onto. Since $\phi^* : k(Y) \to k(X)$ is injective, $\phi^* : k[Y]_s \to k[X]_{\phi^*(s)}$ is injective. Hence $\phi : X_{\phi^*(s)} \to Y_s$ is an isomorphism. This implies one direction of the last assertion, taking $U = Y_s$. The other direction is easy.

Q.E.D.

Lemma 0.18. *Let X be a normal affine variety and let $a \in k[X]$ be nonzero. Then X_a is normal.*

Proof : This follows from the following assertion: let A be an integral domain and let B be the integral closure of A in the fraction field of A. Let $S \subset A$ be a multiplicatively closed set and assume $0 \notin S$. Then $S^{-1}B$ is the integral closure of $S^{-1}A$ in the fraction field of A. This can be proved just as in the proof of (1) implies (2) in Proposition 39, p. 687, of Dummit and Foote.

Q.E.D.

Proposition 0.19. *Let X be an affine variety. There exists nonzero $a \in k[X]$ such that X_a is normal.*

EXERCISE: Prove this. The following steps are useful.

(i) Let B be the integral closure of $k[X]$. Show that B is a finitely generated integrally closed k-algebra (use a commutative algebra result from these notes).

(ii) Let Y be the affine variety such that $k[Y] = B$. Show that there exists a birational morphism $\phi : Y \to X$ such that $\phi^* : k[X] \to B$ is the inclusion.

(iii) Find $a \in k[X]$ such that $\phi : Y_{\phi^*(a)} \to X_a$ is an isomorphism. Conclude that X_a is normal.

Remark 0.20. *Let $\phi : X \to Y$ be a dominant morphism of affine varieties and let $r = \dim(X) - \dim(Y)$. By the proof of Proposition 3.26 in the book "An Introduction to Invariants and Moduli" by Mukai, the author shows that there exists $a \in k[Y]$ and $\eta : X_{\phi^*(a)} \to k^r$ such that the morphism $\psi = (\phi, \eta) : X_{\phi^*(a)} \to Y_a \times k^r$ is finite and dominant, where $r = \dim(X) - \dim(Y)$. By Proposition 0.15, it follows that $\dim(X_{\phi^*(a)}) \geq \dim(Y_a)$, so $r \geq 0$. By Proposition 0.19, there is nonzero $f \in k[Y]$ such that Y_f is normal. Hence, $Y_{af} = (Y_f)_a$ is normal by Lemma 0.18, so $Y_{af} \times k^r$ is normal by 0.7. Let $b = af$. Since localization preserves finiteness, it follows that $\psi : X_{\phi^*(b)} \to Y_b \times k^r$ is a finite dominant morphism to a normal variety.*

Remark 0.21. *(EXERCISE) Prove the following. Let Y be a closed subset of a variety X, and let $Y = Y_1 \cup \cdots \cup Y_k$ be the decomposition of Y into irreducible components. If $V \subset Y$ is an nonempty open set, then the irreducible components of $Y \cap V$ are the $Y_i \cap V$ such that $Y_i \cap V$ is nonempty. Further, if $Y_i \cap V$ is nonempty, then $\dim(Y_i) = \dim(Y_i \cap V)$.*

Theorem 0.22. *Let $\phi : X \to Y$ be a dominant morphism of affine varieties. Let $r = \dim(X) - \dim(Y)$. There exists a nonempty open set U of Y such that if W is a closed subvariety of Y such that $W \cap U$ is nonempty, then for any irreducible component Z of $\phi^{-1}(W \cap U)$, $\dim(Z) = \dim(W) + r$.*

Proof : Take $U = Y_b$ as in Remark 0.20 and let $V = \phi^{-1}(U)$, so U and V are affine varieties. As in Remark 0.20, there exists a morphism $\eta : V \to k^r$ such that $\psi = (\phi, \eta) : V \to U \times k^r$ is a finite dominant morphism to a normal variety. By hypothesis, $(W \cap U) \times k^r$ is a closed subvariety of $U \times k^r$. Let Z be an irreducible component of $\phi^{-1}(W)$ such that $Z \cap V$ is nonempty. By Remark 0.21, $Z \cap V$ is an irreducible component of $\phi^{-1}(W) \cap V = \psi^{-1}((W \cap U) \times k^r)$. Hence, by Proposition 0.15(iv), $\dim(Z \cap V) = \dim((W \cap U) \times k^r) = \dim(W \cap U) + r$. It follows that $\dim(Z) = \dim(W) + r$ by Remark 0.21.

Q.E.D.

Corollary 0.23. *Let $\phi : X \to Y$ be a dominant morphism of affine varieties. There exists a nonempty open set $U \subset \phi(X)$ such that for all $y \in U$ and any irreducible component Z of $\phi^{-1}(y)$, $\dim(Z) = r := \dim(X) - \dim(Y)$.*

Theorem 0.24. *Let $\phi : X \to Y$ be a morphism of affine varieties. For $x \in X$, let $\phi^{-1}\phi(x) = Z_1 \cup \cdots \cup Z_j$ be the irreducible components of $\phi^{-1}\phi(x)$. Let $e(x)$ be the maximum of the dimensions of the Z_i. Let $S_n(\phi) := \{x \in X : e(x) \geq n\}$. Then $S_n(\phi)$ is a closed subset of X.*

Proof : By replacing Y with $\overline{\phi(X)}$, we may assume that ϕ is dominant. Let $r = \dim(X) - \dim(Y) \geq 0$ by Remark 0.20. Argue by induction on $p := \dim(Y)$. If $p = 0$, then $\phi^{-1}(\phi(x)) = \phi^{-1}(Y) = X$ is clearly closed. Assume the assertion is true for $j < p$.

If $n \leq r$, then $S_n(\phi) = X$ by Theorem 0.12, so we may suppose $n > r$. By Corollary 0.23, there exists a nonempty open subset U of Y such that $S_n(\phi) \subset X - \phi^{-1}(U)$. Let $W_1, \ldots, W_s$ be the irreducible components of $Y - U$. Then for all i, $\dim(W_i) < \dim(Y)$ since each W_i is proper and closed. Let $\{Z_{ij}\}_{j \in J_i}$ be the irreducible components of $\phi^{-1}(W_i)$, and let $\phi_{ij} : Z_{ij} \to W_i$ be the restriction of ϕ to Z_{ij}. By the inductive assumption, $S_n(\phi_{ij})$ is closed in Z_{ij}. The reader can check easily that $S_n(\phi) = \cup_{i,j} S_n(\phi_{ij})$, and the result follows.

Q.E.D.