INTEGAL EXTENSIONS

SAM EVENS

1. Introduction

2. Preliminary results

Discussion of integral extensions. We follow closely Samuel, Algebraic Theory of Numbers, section 2.1.

Let R be a subring of a ring S.

Definition 2.1. An element $\alpha \in S$ is called integral over R if there exists a monic polynomial $p(x) \in R[x]$ such that $p(\alpha) = 0$.

For $\alpha_1, \dots, \alpha_n$ as above, $R[\alpha_1, \dots, \alpha_n]$ is by definition the subring of S generated by R and $\{\alpha_1, \dots, \alpha_n\}$. It is not difficult to check that $R[\alpha] = \{\sum r_i \alpha^i : r_i \in R\}$.

Lemma 2.2. *Let $R \subset S$ be as above and let N be a S-module. If S is a finitely generated R-module, and N is a finitely generated S-module, then N is a finitely generated R-module.*

The proof of this last assertion is quite easy.

Theorem 2.3. *Let $\alpha \in S$ with $R \subset S$ as above. The following are equivalent:*

(i) α is integral over R.

(ii) $R[\alpha]$ is a finitely generated R-module.

(iii) There exists a subring T of S such that $R[\alpha] \subset T$ and T is a finitely generated R-module.

We proved this in class. It is Prop 23 in 15.3 of Dummit and Foote.

Proposition 2.4. *Let $R \subset S$ be as above. Let $\alpha_1, \dots, \alpha_n$ be elements of S such that if we let $R_i := R[\alpha_1, \dots, \alpha_i]$, then α_{i+1} is integral over R_i. Then $R[\alpha_1, \dots, \alpha_n]$ is a finitely generated R-module and is integral over R.*

Proof. We prove the assertion by induction on k, and note the assertion is trivial for $R_0 := R$. Since α_{k+1} is integral over R_k, then by Theorem 2.3, $R_{k+1} = R_k[\alpha_{k+1}]$ is a finitely generated R_k-module. By induction, R_k is a finitely generated R-module, so by Lemma 2.2, R_{k+1} is a finitely generated R-module. This proves that $R[\alpha_1, \dots, \alpha_n]$

is a finitely generated R-module. For the last assertion, let $\gamma \in R[\alpha_1, \ldots, \alpha_n]$. Then γ is integral over R by (iii) implies (i) of Theorem 2.3. □

Corollary 2.5. *For $\alpha_1, \ldots, \alpha_n \in S$ which are each integral over R, then $R[\alpha_1, \ldots, \alpha_n]$ is integral over R, and is a finitely generated R-module.*

Proof. The assertion is an easy special case of the previous proposition. □

Corollary 2.6. *Let $R \subset S$ be as above, and let $\overline{R}$ be the set of elements of S which are integral over R. Then $\overline{R}$ is a subring of S.*

Proof. For $\alpha, \beta \in \overline{R}$, then by Proposition 2.4, $R[\alpha, \beta]$ is integral over R. Hence, $\alpha - \beta$ and $\alpha\beta$ are integral over R. □

Proposition 2.7. *Let R, S, and T be rings, with R a subring of S, and S a subring of T. If S is integral over R and T is integral over S, then T is integral over R.*

Proof. Let $\alpha \in T$. Since T is integral over S, there exists $p(x) = s_0 + s_1 x + \ldots s_{n-1} x^{n-1} + x^n \in S[x]$ such that $p(\alpha) = 0$. It follows that α is integral over $R' := R[s_0, \ldots, s_{n-1}]$. By Corollary 2.5, the ring R' is finitely generated as a R-module. By Proposition 2.4, $R'[\alpha]$ is finitely generated over R'. Hence by lemma 2.2, it follows that $R'[\alpha]$ is a finitely generated R-module. Hence by (iii) implies (i) of Theorem 2.3, it follows that α is integral over R. This proves the assertion. □

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556
E-mail address: sevens@nd.edu