HANDOUT ON TRANSCENDENCE DEGREE

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We prove properties of transcendence degree.

Let E/F be a field extension. An element $\alpha \in E/F$ is called transcendental if α is not algebraic over F.

A subset S of E is called *algebraically independent* over F if for every nonempty finite subset $\{\alpha_1, \ldots, \alpha_n\} \subset S$, there is no nonzero polynomial $f = f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$ such that $f(\alpha_1, \ldots, \alpha_n) = 0$. It follows that the empty set is algebraically independent. If S is not algebraically independent over F, S is called *algebraically dependent* over F.

Note that a subset of an algebraically independent set is trivially algebraically independent.

Definition 1. *If E/F is a field extension, a subset S of E is called a transcendence basis of E/F if S is algebraically independent over F and $E/F(S)$ is algebraic.*

If E is algebraic over $F(S)$, we say that S spans E algebraically over F. Thus, a transcendence basis is an algebraically independent set over F spanning E algebraically over F. The terminology suggests the close analogy between the notion of transcendence basis and linear basis of a vector space over a field F.

The main results to prove are:

Theorem 2 (Ash, 6.9.3). *If E/F is a field extension, there is a transcendence basis of E over F.*

Theorem 3 (Ash, 6.9.5). *Let E/F be a field extension with transcendence bases S and T. Then S is finite if and only if T is finite, and if so, $|S| = |T|$.*

More generally, if S and T are two transcendence bases of E over F, then $|S| = |T|$. Theorem 3 above suffices for most applications in algebra.

Definition. *Let E/F be a field extension with transcendence basis S. Then the transcendence degree of E/F is $|S|$, which is independent of the choice of S by the above remarks.*

Let F be a field and let $f = f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$ be nonzero. We say f depends on x_1 if f is not in $F[x_2, \ldots, x_n]$, viewed as a subring of $F[x_1, \ldots, x_n]$ in the obvious way.

Lemma 4. *Let E/F be a field extension and let $S \subset E$ be algebraically independent over F. Let $\alpha \in E - S$. Then α is algebraic over $F(S)$ if and only if $S \cup \{\alpha\}$ is algebraically dependent over F.*

The analogous assertion in linear algebra is that if V is a vector space over a field F and $S \subset V$ is linearly independent and $v \in V$ is not in S, then v is in the F-span of S if and only if $S \cup \{v\}$ is linearly dependent. This linear algebra result is essentially trivial. Lemma 4 is treated as a triviality in Ash, section 6.9. While the proof of Lemma 4 is not difficult, it is nontrivial, and it is useful to make it explicit.

Proof of Lemma 4. If α is algebraic over $F(S)$, there are $c_{k-1}, \ldots, c_0 \in F(S)$ so that

$\alpha^k + c_{k-1}\alpha^{k-1} + c_1\alpha + c_0 = 0$

It is easy to check that $F(S)$ is the fraction field of $F[S]$, the smallest subring of E containing F and S. Thus, each $c_i = \frac{a_i}{b_i}$, with $a_i, b_i \in F[S]$ and b_i nonzero. Then $d := \Pi_{i=0}^{k-1} b_i \in F[S]$ is nonzero.

Let $g(y) = dy^k + dc_{k-1}y^{k-1} + \cdots + dc_0 \in F[S][y]$. Then $g(y)$ is a nonzero polynomial, and $g(\alpha) = 0$.

Since every element γ of $F[S]$ is in $F[C]$, where $C \subset S$ is a finite subset depending on γ, it follows that the elements d and $dc_i, i = 0, \ldots, k-1$, are in $F[W]$ for some subset $W = \{\beta_1, \ldots, \beta_s\} \subset S$.

Let $R = F[x_1, \ldots, x_s]$ and use the universal property of polynomial rings to define a surjective ring homomorphism $\chi : R \to F[W]$ by $\chi(x_i) = \beta_i$ for all i and $\chi|_F = id_F$. Identify $R[y] = F[y, x_1, \ldots, x_s]$, and extend χ to a surjective ring homomorphism $\chi : R[y] \to F[W][y]$ by $\chi(\sum a_i y^i) = \sum \chi(a_i)y^i$ (the a_i are in R). It follows from the definition that if $f \in R[y] = F[y, x_1, \ldots, x_s]$, and $\gamma \in F$, then $\chi(f)(\gamma) = f(\gamma, \beta_1, \ldots, \beta_m)$.

Thus, there is nonzero $f \in R[y] = F[y, x_1, \ldots, x_s]$ such that $\chi(f(y)) = g(y)$, so $f(\alpha, \beta_1, \ldots, \beta_s) = \chi(f)(\alpha) = g(\alpha) = 0$. Hence, $S \cup \{\alpha\}$ is algebraically dependent over F.

Conversely, assume $S \cup \{\alpha\}$ is algebraically dependent over F, so there exists a subset $W = \{\beta_1, \ldots, \beta_s\} \subset S$ and a nonzero polynomial $f \in F[y, x_1, \ldots, x_s]$ such that $f(\alpha, \beta_1, \ldots, \beta_s) = 0$. Let $R = F[y]$ as above, and $R[x_1, \ldots, x_s] = F[y, x_1, \ldots, x_s]$, and define $\chi : F[y, x_1, \ldots, x_s] \to F[y]$ by mapping x_i to β_i as above. Then f depends on y since W is algebraically independent over F, as it is a subset of S. Thus, we may write

$f = f(y) = a_k y^k + \cdots + a_1 y + a_0 \in R[y]$ with $k > 0$ and $a_k \neq 0$. As above, $\chi(f)(\alpha) = f(\alpha, \beta_1, \ldots, \beta_s) = 0$.

Let $d = \chi(a_k)$. Then d is nonzero since a_k is nonzero and W is algebraically independent.

Set $g(y) = \frac{\chi(f)(y)}{d} = y^k + \frac{\chi(a_{k-1})}{d} y^{k-1} + \cdots + \frac{\chi(a_0)}{d}$.

Then $g(\alpha) = \frac{\chi(f)(\alpha)}{d} = 0$, so α is algebraic over $F(W)$ and hence over $F(S)$.

$\square$

Remark 5. *For a field extension E/F, let $S = \{\alpha_1, \ldots, \alpha_n\} \subset E$ with $S_{\alpha_1} = S - \{\alpha_1\}$ algebraically independent over F. Suppose there is a nonzero polynomial $f = f(x_1, \ldots, x_n) \in F[x_1, \ldots, x_n]$ such that $f(\alpha_1, \ldots, \alpha_n) = 0$ and f depends on x_1. Then α_1 is algebraic over $F(\alpha_2, \ldots, \alpha_n)$. Indeed, this may be proved by repeating the proof of the converse assertion in Lemma 4, letting $\alpha = \alpha_1$ and letting $\{\alpha_2, \ldots, \alpha_n\}$ play the role of W.*

Lemma 6. *Let E/F be a field extension and let S span E algebraically over F. For $\alpha \in S$, let $S_\alpha = S - \{\alpha\}$. Then S_α spans E algebraically over F if and only if α is algebraic over $F(S_\alpha)$.*

Proof of Lemma 6. If α is algebraic over $F(S_\alpha)$, then $F(S)$ is algebraic over $F(S_\alpha)$. Since E is algebraic over $F(S)$, it follows by Ash, Corollary, 3.3.5, that E is algebraic over $F(S_\alpha)$. For the converse, if $E/F(S_\alpha)$ is algebraic, then certainly α is algebraic over $F(S_\alpha)$. $\square$

Proof of Theorem 2. Let V be the collection of subsets U of E such that U is algebraically independent over F. If U_i, U_j are in V, we say $U_i \le U_j$ if $U_i \subset U_j$. V is nonempty since the empty set is in V, so V is a nonempty poset.

Let $A \subset V$ be a totally ordered subset. Let $U_A = \cup_{U_i \in A} U_i$. Then U_A is algebraically independent over F. Indeed, if $W = \{\alpha_1, \ldots, \alpha_n\}$ is a subset of U_A, then $\alpha_j \in U_{i_j}$ for some $U_{i_j} \in A$, so since A is totally ordered, there is $k, 1 \le k \le n$ so that $U_{i_j} \le U_{i_k}$ for all $j, 1 \le j \le n$. Thus, all α_j are in U_{i_k}, so since U_{i_k} is algebraically independent over F, W is algebraically independent over F. Since $U_i \subset U_A$ for all $U_i \in A$, U_A is an upper bound for A. Thus, the hypotheses of Zorn's Lemma are satisfied, so E has a subset S such that S is maximal among all algebraically independent subsets over F.

We claim that S is a transcendence basis of E over F. Indeed, if $\alpha \in E - S$, then $S \cup \{\alpha\}$ is algebraically dependent by maximality of S, so α is algebraic over $F(S)$ by Lemma 4. $\square$

Proof of Theorem 3. We prove that if $|T|$ is finite, then $|S| \le |T|$. Switching roles of S and T, it follows that if $|S|$ is finite, then $|T| \le |S|$, which implies the result.

Let $|T| = m$. If $|S| > m$, there is a subset $\{\alpha_1, \ldots, \alpha_{m+1}\}$ of S that is algebraically independent over F.

Let $S_0 = T$. We show by induction on i that there exists an ordering $T = \{\beta_1, \ldots, \beta_m\}$ such that if

$(***)$ $S_i = \{\alpha_1, \ldots, \alpha_i, \beta_{i+1}, \ldots, \beta_m\}$,

then S_i is a transcendence basis of E over F for $i = 0, \ldots, m$. Given this, $S_m = \{\alpha_1, \ldots, \alpha_m\}$ is a transcendence basis of E/F, so α_{m+1} is algebraic over $F(\{\alpha_1, \ldots, \alpha_m\})$. Thus by Lemma 4, S is algebraically dependent over F. This is a contradiction, so $|S| \le |T|$.

The assertion is clear if $i = 0$, so we assume $i > 0$ and we have found $\beta_1, \ldots, \beta_{i-1}$ in T so S_{i-1} as defined in (***) is a transcendence basis of E/F. We now find β_i in T so S_i is a transcendence basis.

Since $E/F(S_{i-1})$ is algebraic, α_i is algebraic over $F(S_{i-1})$. By Lemma 4, $\{\alpha_i\} \cup S_{i-1}$ is algebraically dependent over F, so there is a nonzero polynomial $f(y_1, \ldots, y_i, x_i, \ldots, x_m) \in F[y_1, \ldots, y_i, x_i, \ldots, x_m]$ with $f(\alpha_1, \alpha_i, \beta_i, \ldots, \beta_m) = 0$ (here, the y_j and the x_k are all variables, and we use different letters for them because y_j evaluates to α_j and x_k evaluates to β_k).

Choose a nonzero polynomial $h \in F[y_1, \ldots, y_i, x_i, \ldots, x_m]$ such that h depends only on exactly r of the variables x_k with r minimal and $h(\alpha_1, \ldots, \alpha_i, \beta_i, \ldots, \beta_m) = 0$. Then h must depend on some x_k with $k \geq i$. Indeed if not,

$$h(\alpha_1, \ldots, \alpha_i, \beta_i, \ldots, \beta_m) = 0$$

makes $\{\alpha_1, \ldots, \alpha_i\}$ algebraically dependent over F. By renumbering, we may assume that h depends on $x_i, x_{i+1}, \ldots, x_{i+r-1}$.

We claim further that $\{\alpha_1, \ldots, \alpha_i, \beta_{i+1}, \ldots, \beta_{i+r-1}\}$ is algebraically independent over F. Indeed, if they are algebraically dependent, there is a nonzero polynomial h_1 depending on fewer than $r - 1$ variables from the set $\{x_i, \ldots, x_m\}$ so that $h_1(\alpha_1, \ldots, \alpha_i, \beta_i, \ldots, \beta_m) = 0$, which contradicts the minimality of r.

Thus, by Remark 5, β_i is algebraic over $S_i = \{\alpha_1, \ldots, \alpha_i, \beta_{i+1}, \ldots, \beta_m\}$. But $V = S_i \cup \{\beta_i\}$ contains S_{i-1}, so E is algebraic over $F(V)$, so by Lemma 6 with $\alpha = \beta_i$, E is algebraic over $F(S_i)$.

It remains to prove that S_i is algebraically independent over F. Note that $U_i = S_{i-1} - \{\beta_i\}$ is algebraically independent over F, since it is a subset of the algebraically independent set S_{i-1}. Since $S_i = U_i \cup \{\alpha_i\}$, it follows that if S_i is algebraically dependent over F, then by Lemma 4, α_i is algebraic over $F(U_i)$. But we just checked that β_i is algebraic over $F(S_i)$, so by Ash, Cor. 3.3.5, it follows that β_i is algebraic over $F(U_i)$, so S_{i-1} is algebraically dependent over F by Lemma 4 again. But S_{i-1} is algebraically independent, so S_i is algebraically independent over F. □

Remark. *It is not difficult to prove that if E/F has transcendence degree k and K/E is a field extension of transcendence degree r, then K/F is an extension with transcendence degree $k + r$.*