Regenerative cascade homotopies for solving polynomial systems

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Abstract

A key step in the numerical computation of the irreducible decomposition of a polynomial system is the computation of a *witness superset* of the solution set. In many problems involving a solution set of a polynomial system, the witness superset contains all the needed information. Sommese and Wampler gave the first numerical method to compute witness supersets, based on dimension-by-dimension slicing of the solution set by generic linear spaces, followed later by the cascade homotopy of Sommese and Verschelde. Recently, the authors of this article introduced a new method, *regeneration*, to compute solution sets of polynomial systems. Tests showed that combining regeneration with the dimension-by-dimension algorithm was significantly faster than naively combining it with the cascade homotopy. However, in this article, we combine an appropriate randomization of the polynomial system with the regeneration technique to construct a new cascade of homotopies for computing witness supersets. This *regenerative cascade* is superior in practice to all known methods.

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Introduction

Numerical algebraic geometry is the computation and manipulation of the solution sets of systems of polynomials using numerical algorithms (see [13] for a comprehensive development of the area). Let

$$ f(z) = \begin{bmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{bmatrix} = 0 \tag{1} $$

denote a system of n polynomials with $z \in \mathbb{C}^N$. The basic data structure to describe an irreducible component X of the solution set

$$ V(f) := \left\{ z \in \mathbb{C}^N \mid f(z) = 0 \right\} $$

of this system is a *witness set* for X. When X is a multiplicity one irreducible component of $V(f)$, a witness set of X is a triple $(f, \mathcal{L}, X \cap \mathcal{L})$ consisting of the polynomial system f, a random affine linear subspace $\mathcal{L}$ of $\mathbb{C}^N$ of dimension complementary to the dimension of X, plus the points $X \cap \mathcal{L}$. For a component of multiplicity greater than one, a witness set contains extra information that is computable from the triple $(f, \mathcal{L}, X \cap \mathcal{L})$ [13, §13.3]. In particular, the triple $(f, \mathcal{L}, X \cap \mathcal{L})$ is the key object to compute, and for practical purposes, may be regarded as the witness set of the irreducible component X.

   Witness sets are a natural data structure to numerically describe the decomposition of a solution set into irreducible components [8, 13] and to enable the computation of other detailed information about algebraic sets, e.g., their intersections [9, 10]. Using homotopy continuation, it is computationally inexpensive to compute as many random and widely-distributed points on X as desired, from which, for instance, polynomials cutting out the component X may be computed.

   A witness set of X is computed from a witness superset of X. A *witness superset* of X is a triple $(f, \mathcal{L}, S)$ where f and $\mathcal{L}$ are as above and S is a

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finite set of points with $(X \cap \mathcal{L}) \subset S \subset (V(f) \cap \mathcal{L})$. The points in $S \setminus X$ are called *junk points* and are filtered out to create a witness set.

The numerical irreducible decomposition of $V(f)$ is a collection, say $\mathcal{W}$, of witness sets for each irreducible component of $V(f)$. The computation of $\mathcal{W}$ proceeds in three steps

1. computation of a witness superset $\widehat{\mathcal{W}}$ of $V(f)$;

2. computation of a witness set $\mathcal{W}$ of $V(f)$ from $\widehat{\mathcal{W}}$;

3. decomposition of the witness set $\mathcal{W}$ into witness sets of the irreducible components of $V(f)$.

There are two existing algorithms to perform Step 1, namely a dimension-by-dimension algorithm presented in [12] and a cascade algorithm presented in [7]. While the computational cost of the two approaches is usually roughly comparable, these methods may differ in the number of junk points they produce. In this way, the choice of method directly affects the computational cost of Step 2. In this regard, the cascade algorithm is favored as it generally computes many fewer junk points than the dimension-by-dimension algorithm.

It should be mentioned that a version of the regeneration algorithm presented in [3] performs Steps 1 and 2 together in an integrated way. A significant drawback of that version of regeneration is that it may require deflation of intermediate systems that arise. While a methodology exists for this, it is complicated to implement. Neither the dimension-by-dimension method nor the cascade method have this drawback. Like these, the new algorithm presented in this paper also does not require deflation. It is based on the version of regeneration from [3] that seeks only to find isolated solution points. In this article, we limit our computational experiments to the three methods that never require deflation.

Step 3 is performed using a monodromy algorithm that is certified using a trace test [11]. This step is independent of Steps 1 and 2, so we shall say nothing further about it here.

For many problems, Step 1 is the most computationally intensive part of finding a numerical irreducible decomposition. The subject of this article, the regenerative cascade, addresses this step. We provide evidence that the new approach matches the efficiency of the older cascade algorithm in regards to the number of junk points it produces while reducing the cost of Step 1 as compared to both of the prior algorithms.

This article is organized as follows. Section 1 provides an overview of regeneration [3]. Section 2 presents the regenerative cascade algorithm; Section 2.3 describes the advantages of the regenerative cascade algorithm; and section 3 provides computational results.

1 Regeneration

Before describing the regenerative cascade algorithm in § 2, we summarize the regeneration algorithm [3] for computing the nonsingular isolated solutions of a square polynomial system utilizing an equation-by-equation approach with a general linear product decomposition.

Let

$$f(z) = f(z_1, \ldots, z_n) = \begin{bmatrix} f_1(z_1, \ldots, z_n) \\ \vdots \\ f_n(z_1, \ldots, z_n) \end{bmatrix}$$

be a polynomial system with $d_i = \deg f_i$. For $i = 1, \ldots, n$ and $j = 1, \ldots, d_i$, let $L_{i,j}(z)$ be a general linear function on $\mathbb{C}[z_1, \ldots, z_n]$ and, for $m = 0, \ldots, n$, define

$$F_m(z) = \begin{bmatrix} f_1(z) \\ \vdots \\ f_m(z) \\ L_{m+1,1}(z) \\ L_{m+2,1}(z) \\ \vdots \\ L_{n,1}(z) \end{bmatrix} \quad \text{and} \quad G_m(z) = \begin{bmatrix} f_1(z) \\ \vdots \\ f_m(z) \\ \prod_{j=1}^{d_{m+1}} L_{m+1,j}(z) \\ L_{m+2,1}(z) \\ \vdots \\ L_{n,1}(z) \end{bmatrix}.$$

Regeneration starts with the solution of the linear system

$$F_0(z) = \begin{bmatrix} L_{1,1}(z) \\ \vdots \\ L_{n,1}(z) \end{bmatrix} = 0$$

and computes the solutions of $F_n(z) = f(z) = 0$ by using a sequence of two types of homotopies. For $0 \leq m \leq n-1$ and $1 \leq a \leq d_i$, the first type of homotopy is a parameter homotopy that moves from the linear space

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$L_{m,1} = 0$ to $L_{m,a} = 0$, namely

$$H_{m,a}^{\text{parm}}(z,t) = \begin{bmatrix} f_1(z) \\ \vdots \\ f_m(z) \\ (1-t)L_{m+1,a}(z) + tL_{m+1,1}(z) \\ L_{m+2,1}(z) \\ \vdots \\ L_{n,1}(z) \end{bmatrix}.$$

For $0 \le m \le n-1$, the second type of homotopy is a general linear product homotopy that moves from the product of linear spaces $\prod_{i=1}^{d_{m+1}} L_{m+1,i} = 0$ to $f_{m+1} = 0$, namely

$$H_m^{\text{prod}}(z,t) = (1-t)F_{m+1}(z) + tG_m(z)$$

$$= \begin{bmatrix} f_1(z) \\ \vdots \\ f_m(z) \\ (1-t)f_{m+1}(z) + t\prod_{i=1}^{d_{m+1}} L_{m+1,i}(z) \\ L_{m+2,1}(z) \\ \vdots \\ L_{n,1}(z) \end{bmatrix}.$$

The following algorithm computes the nonsingular isolated points of $V(f)$ using regeneration.

Procedure $S = \mathbf{Regenerate}(f)$

Input A set $f = \{f_1, \ldots, f_n\}$ of n polynomials on $\mathbb{C}^n$.

Output A set $S \subset \mathbb{C}^n$ consisting of the nonsingular isolated points of $V(f)$.

Begin 1. Let $d_i = \deg f_i$. For $i = 1, \ldots, n$ and $j = 1, \ldots, d_i$, let $L_{i,j}(z)$ be a random linear function on $\mathbb{C}[z_1, \ldots, z_n]$.

2. Use numerical linear algebra to compute the set S_0 consisting of the solution of the linear system $F_0 = 0$.

3. For $m = 0, \ldots, n-1$, do the following:

(a) Solve for T_m, the set of nonsingular isolated points of $V(G_m)$, using the homotopies $H_{m,j}^{\text{parm}}$, $j = 1, \ldots, d_{m+1}$, with start points S_m.

(b) Solve for S_{m+1}, a superset of the nonsingular isolated points of $V(F_{m+1})$, using the homotopy H_m^{prod} with start points T_m.

(c) Expunge any singular points in S_{m+1}.

Return $S = S_n$.

Note 1.1 $H_{m,1}^{\text{parm}}(z,t) \equiv F_m(z)$.

2 Regenerative cascade

The regenerative cascade algorithm is applicable to the following basic problem in numerical algebraic geometry.

Problem 1 (Witness Superset) *Given a polynomial system $f : \mathbb{C}^N \to \mathbb{C}^n$ find a witness superset for $V(f)$.*

2.1 Regenerative cascade algorithm

Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system as in Eq. 1 with $d_1 \geq \cdots \geq d_n$ where $d_i = \deg f_i$. Let $r = \text{rank}(f)$ (see [13, § 13.4]) and $\alpha_{i,j} \in \mathbb{C}$ be random for $1 \leq i \leq r$ and $i < j \leq n$. Define

$$
A = \begin{bmatrix}
1 & \alpha_{1,2} & \alpha_{1,3} & \cdots & \cdots & \cdots & \alpha_{1,n} \\
 & 1 & \alpha_{2,3} & \cdots & \cdots & \cdots & \alpha_{2,n} \\
 & & \ddots & & & & \vdots \\
 & & & 1 & \alpha_{r,r+1} & \cdots & \alpha_{r,n}
\end{bmatrix} \tag{2}
$$

and

$$
\widehat{f} = A \cdot f. \tag{3}
$$

The polynomial system $\widehat{f}$ consists of r polynomials in N variables with $r \leq N$ and $\deg \widehat{f}_i = d_i$. Since $\dim V(\widehat{f}) \geq N - r$, we will append $N - r$ general linear functions onto $\widehat{f}$. That is, for $i = 1, \ldots, N - r$, let $\mathcal{L}_i(z)$ be a generic linear function on $\mathbb{C}[z_1, \ldots, z_N]$ and $\mathcal{L} = \begin{bmatrix} \mathcal{L}_1 \\ \vdots \\ \mathcal{L}_{N-r} \end{bmatrix}$. Define

$$
\mathcal{F} = \begin{bmatrix} \mathcal{L} \\ \widehat{f} \end{bmatrix}. \tag{4}
$$

The following lemma shows how to compute a witness superset for $V(f)$ using a witness superset for $V(\mathcal{F})$.

In the above, "generic" means that the generality needed to obtain the results we seek holds for all but a proper algebraic subset of the set of all possible choices of $\alpha_{i,j}$ and the coefficients defining the linear functions $\mathcal{L}$ and L_i, $i = 1, \ldots, N - r$. Let us call the complex Euclidean space of all these parameters the "algorithm parameter space."

Lemma 2.1 *For a Zariski open dense subset of the algorithm parameter space, the following holds. Let $\mathcal{W}_i = \{\mathcal{F}, L_i, \mathcal{X}_i\}$ be a witness superset for the ith dimensional irreducible components of $V(\mathcal{F})$. If $X_i = \{x \in \mathcal{X}_i \mid f(x) = 0\}$, then $W_{N-r+i} = \{f, \{\mathcal{L}, L_i\}, X_i\}$ is a witness superset for the $(N-r+i)$-th dimensional irreducible components of $V(f)$.*

Proof. Altogether $\{\mathcal{L}, L_i\}$ are $N - r + i$ generic linear equations, hence $\mathcal{X}_i$ is a witness superset for the $(N - r + i)$-dimensional irreducible components of $\widehat{f}$. By the randomization theorem, Theorem 13.5.1, of [13], all the $(N-r+i)$-dimensional irreducible components of f will be among these, but only points in $V(f)$ can be witness points for components of $V(f)$. $\qquad\square$

For $i = 1, \ldots, N - r$ and $j = 1, \ldots, d_i$, let $L_{i,j}(z)$ be a general linear function on $\mathbb{C}[z_1, \ldots, z_n]$. The following are analogous to $F_m, G_m, H_{m,a}^{\mathrm{parm}}$, and H_m^{prod} defined in § 1:

$$
\mathcal{F}_m(z) = \begin{bmatrix} \mathcal{L}(z) \\ \widehat{f}_1(z) \\ \vdots \\ \widehat{f}_m(z) \\ L_{m+1,1}(z) \\ L_{m+2,1}(z) \\ \vdots \\ L_{r,1}(z) \end{bmatrix}, \quad
\mathcal{G}_m(z) = \begin{bmatrix} \mathcal{L}(z) \\ \widehat{f}_1(z) \\ \vdots \\ \widehat{f}_m(z) \\ \prod_{j=1}^{d_{m+1}} L_{m+1,j}(z) \\ L_{m+2,1}(z) \\ \vdots \\ L_{r,1}(z) \end{bmatrix},
$$

$$
\mathcal{H}_{m,a}^{\mathrm{parm}}(z,t) = \begin{bmatrix} \mathcal{L}(z) \\ \widehat{f}_1(z) \\ \vdots \\ \widehat{f}_m(z) \\ (1-t)L_{m+1,a}(z) + tL_{m+1,1}(z) \\ L_{m+2,1}(z) \\ \vdots \\ L_{r,1}(z) \end{bmatrix},
$$

and $\mathcal{H}_m^{\mathrm{prod}}(z,t) = (1-t)\mathcal{F}_{m+1}(z) + t\mathcal{G}_m(z)$.

A point $x \in V(\mathcal{F}_m)$ (or $V(\mathcal{G}_m)$) is a *nonsolution* with respect to f if $f(x) \neq 0$. Bertini's theorem and genericity provide that nonsolutions of $\mathcal{F}_m$ and $\mathcal{G}_m$ are nonsingular isolated solutions of $\mathcal{F}_m$ and $\mathcal{G}_m$, respectively. The following lemma shows that regenerating the nonsolutions of $\mathcal{F}_m$ will yield a superset of the isolated solutions of $\mathcal{F}_{m+1}$.

Lemma 2.2 *For a Zariski open dense subset of the algorithm parameter space, the set of solutions of $\mathcal{F}_{m+1}$ obtained by regenerating the nonsolutions of $\mathcal{F}_m$ contains the isolated solutions of $\mathcal{F}_{m+1}$.*

Proof. According to the theory underlying the regeneration algorithm (§ 1), the isolated solutions of $\mathcal{F}_{m+1}$ are contained in the set of endpoints of $\mathcal{H}_m^{\mathrm{prod}}$ using the isolated solutions of $\mathcal{G}_m$ as start points, which themselves are obtained by the homotopies $\mathcal{H}_{m,j}^{\mathrm{parm}}$, $j = 1, \ldots, d_{m+1}$ using the isolated solutions of $\mathcal{F}_m$ as start points. If, for $m < r$, x is an isolated solution of $\mathcal{F}_m$ with $f(x) = 0$, then it lies on a component of $V(f)$ of dimension at least m. The paths from such a point must remain on this component during the homotopies $\mathcal{H}_m^{\mathrm{prod}}$ and $\mathcal{H}_{m,j}^{\mathrm{parm}}$, $j = 1, \ldots, d_{m+1}$. and hence cannot lead to nonsingular solutions to $\mathcal{F}_{m+1}$. Thus, the paths originating from the nonsolutions of $\mathcal{F}_m$ suffice to find all isolated solutions of $\mathcal{F}_{m+1}$. $\qquad \square$

Procedure $S = $ **RegenerativeCascade**(f)

Inputs A set $f = \{f_1, \ldots, f_n\}$ of n polynomials on $\mathbb{C}^N$.

Output A witness superset S for $V(f)$.

Begin   1. Rename f so that $d_i \geq \cdots \geq d_n$ where $d_i = \deg f_i$ and compute $r = \mathrm{rank}(f)$.

           2. For $1 \leq i \leq r$ and $i < j \leq n$, let $\alpha_{i,j} \in \mathbb{C}$ be random. Construct A and $\widehat{f}$ as in Eqs. 2 and 3, respectively.

           3. Let $\mathcal{L}(z)$ be a set of $N-r$ random linear functions on $\mathbb{C}[z_1, \ldots, z_N]$ and construct $\mathcal{F}$ as in Eq. 4.

           4. For $i = 1, \ldots, r$ and $j = 1, \ldots, d_i$, let $L_{i,j}(z)$ be a random linear function on $\mathbb{C}[z_1, \ldots, z_N]$.

           5. Use numerical linear algebra to compute the set X_0 consisting of the solution of the linear system $\mathcal{F}_0 = 0$.

           6. For $m = 0, \ldots, r - 1$, do the following:

(a) Solve for T_m, the set of nonsolutions with respect of f in $V(\mathcal{G}_m)$, using the homotopies $\mathcal{H}^{\mathrm{parm}}_{m,j}$, $j = 1, \ldots, d_{m+1}$, with start points X_m.

(b) Solve for U_{m+1}, a set containing the nonsolutions with respect to f and the isolated points in $V(\mathcal{F}_{m+1})$, using the homotopy $\mathcal{H}^{\mathrm{prod}}_m$ with start points T_m.

(c) Let $X_{m+1} \subset U_{m+1}$ be the nonsolutions of f and $S_{m+1} = \{f, \{\mathcal{L}, L_{m+2,1}, \ldots, L_{r,1}\}, U_{m+1} \setminus X_{m+1}\}$.

Return $S = \{S_1, \ldots, S_r\}$.

The following theorem justifies the regenerative cascade algorithm **RegenerativeCascade**.

Theorem 2.3 *For a Zariski open dense subset of the algorithm parameter space, the* **RegenerativeCascade** *solves Problem 1.*

Proof. This is a straightforward consequence of Lemmas 2.1 and 2.2. $\square$

2.2 Extrinsic vs. intrinsic slicing

Similar to [3, §6.3], the homotopies $\mathcal{H}^{\mathrm{parm}}$ and $\mathcal{H}^{\mathrm{prod}}$ are called extrinsic regenerative cascade homotopies due to the presence of extrinsic linear slicing functions. Since the linear functions $\mathcal{L}, L_{m+2,1}, \ldots, L_{r,1}$ do not change during the path tracking for these homotopies, we may use an intrinsic formulation that is more efficient when $m \ll N$. Numerical linear algebra can be used to compute $B \in \mathbb{C}^{N \times (m+1)}, b \in \mathbb{C}^N$ such that rank $B = m + 1$ and for all $u \in \mathbb{C}^{m+1}$, $Bu + b \in V(\mathcal{L}, L_{m+2,1}, \ldots, L_{r,1})$. The homotopies $\mathcal{H}^{\mathrm{parm}}$ and $\mathcal{H}^{\mathrm{prod}}$ can then be replaced with ones of the form $\widehat{H}(u, t) = H(Bu + b, t)$. Since the linear functions are always zero and can be dropped, these new homotopies consist of $m + 1$ functions and variables, instead of N.

For efficiency, the polynomials should be evaluated in a straight-line manner rather than expanded in the new variables. When the intrinsic formulation is heuristically advantageous, Bertini [1] automatically utilizes it.

2.3 Advantages of the regenerative cascade algorithm

The regenerative cascade algorithm presented in § 2 has several advantages over the the dimension-by-dimension algorithm of [12] and the cascade algorithm of [7]. For a polynomial system $f : \mathbb{C}^n \to \mathbb{C}^n$ of rank n with

$d_1 \geq \cdots \geq d_n$, where $d_i = \deg f_i$, this section describes the theoretical advantages with computational evidence presented in § 3.

The first advantage is that the regenerative cascade is amenable to intrinsic slicing. This reduces the number of variables for tracking the paths, which can reduce the computational costs associated with linear algebra. While the dimension-by-dimension algorithm can also use intrinsic slicing, the original cascade cannot. It uses a homotopy in n variables at every stage.

A second advantage is that the regenerative cascade often tracks fewer paths than either of the other two algorithms. In the case of the original cascade algorithm, it is the first stage of the cascade that is expensive to solve. If one uses a total degree homotopy for this stage, the total degree number of paths, namely $d_1 \cdots d_n$, must be tracked. One could instead use a polyhedral homotopy [4, 6] to possibly reduce the number of paths to track, but this comes at the cost of constructing the polyhedral homotopy, which can be a significant cost to bear. In the case of the dimension-by-dimension algorithm, the last stage is the one that must track the total degree number of paths or use a polyhedral homotopy. Due to the structure of the regenerative cascade algorithm, in which endpoints on higher-dimensional components (including ones at infinity) do not initiate paths at the next stage, the total number of paths that need to be tracked can be less than the total degree.

Finally, the termination of further tracking on all but the nonsolutions generally leads to a witness superset that contains fewer junk points. The regenerative cascade and original cascade algorithms share this advantage. In contrast, the dimension-by-dimension algorithm handles each dimension independently, and so has no mechanism for results obtained at a higher dimension to affect the computation at a lower dimension. This generally leads to a witness superset containing more junk points. Computational evidence, such as Table 2, suggests that the number of junk points of the cascade and regenerative cascade algorithms are related.

Problem 2 *What is the relation between the junk sets in the witness supersets produced by the cascade and regenerative cascade algorithms?*

We do not yet have an answer to this problem, but the results of the following section are suggestive.

	paths tracked (slices moved)			number of junk points		
n	Cascade	Dimension-by-dimension	Regenerative cascade	Cascade	Dimension-by-dimension	Regenerative cascade
3	56	30	26 (12)	6	10	6
4	295	126	96 (47)	30	60	30
5	1,380	510	340 (169)	125	295	125
6	6,050	2,046	1,190 (594)	486	1,342	486
7	25,465	8,190	4,150 (2,074)	1,813	5,853	1,813
8	104,247	32,766	14,456 (7,227)	6,600	24,910	6,600
9	418,289	131,070	50,336 (25,167)	23,665	104,399	23,665
10	1,653,320	524,286	175,246 (87,622)	84,028	433,068	84,028

Table 1: Comparison for computing a witness superset for $P_{2,3,n}$ using various algorithms

3 Computational results

The regenerative cascade algorithm **RegenerativeCascade** is implemented in Bertini [1]. The examples discussed here were run using Bertini v1.2. The results presented in Table 2 were run on a 2.4 GHz Opteron 250 processor with 64-bit Linux. The results presented in Table 3 were run on a cluster consisting of a manager that uses one core of a Xeon 5410 processor and 8 computing nodes each containing two 2.33 GHz quad-core Xeon 5410 processors running 64-bit Linux, i.e., one manager and 64 workers.

3.1 A collection of high-dimensional examples

Consider computing a witness superset for the polynomial system, denoted $P_{2,3,n}$ constructed by taking the 2×2 adjacent permanents of a $3 \times n$ matrix with indeterminant entries [5]. For $n \geq 3$, $V(P_{2,3,n})$ consists of components in multiple dimensions. Table 1 compares the number of paths tracked and the number of junk points in the witness superset for the cascade algorithm, dimension-by-dimension algorithm, and the regenerative cascade algorithm. Table 2 lists the time needed for computing a witness superset for $P_{2,3,n}$, $3 \leq n \leq 9$, using a single processor. Table 3 lists the time needed for computing a witness superset for $P_{2,3,n}$, $8 \leq n \leq 10$, using parallel processing. One may see that the regenerative cascade consistently tracks fewer paths and uses less computation time than the alternatives. Moreover, the two cascade algorithms are equal in the number of junk points generated, beating the dimension-by-dimension method in this regard. Fewer junk points means less work in the next stage of computing a witness set, which is the removal of junk points from the witness superset.

n	Cascade	Dimension-by-dimension	Regenerative cascade
3	0.26s	0.22s	0.19s
4	2.46s	2.20s	1.52s
5	18.8s	17.7s	9.92s
6	2m13s	2m12s	1m9s
7	13m58s	17m25s	6m33s
8	1h22m16s	1h51m8s	36m30s
9	7h30m1s	10h57m17s	2h56m46s

Table 2: Computing a witness superset for $P_{2,3,n}$ using various algorithms

n	Cascade	Dimension-by-dimension	Regenerative cascade
8	2m16s	1m52s	51.4s
9	10m12s	10m50s	3m38s
10	54m3s	60m37s	16m27s

Table 3: Computing a witness superset for $P_{2,3,n}$ using various algorithms in parallel

4 Summary

The regenerative cascade algorithm **RegenerativeCascade** combines the advantages of the cascade, dimension-by-dimension, and regeneration algorithms for computing a witness superset. It utilizes randomization to avoid the disadvantage of the version of regeneration that sometimes needs to deflate systems during the algorithm. Computational evidence suggests that it computes a witness superset with a similar number of junk points as the cascade algorithm, which is generally less than the number of junk points generated by the dimension-by-dimension algorithm. Moreover, in our tests, the regenerative cascade tracks the fewest number of paths, which coupled with the additional advantage of being amenable to intrinsic path tracking, makes it the most efficient method for generating a witness superset.

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