Review No. 1

Review/Overview of the Method of Weighted Residuals

d.e. $L(u) = p(x)$ in Ω

b.c.'s $S(u) = g$ on Γ

Let:

$$u_{app} = u_B + \sum_{k=1}^{N} \alpha_k \phi_k$$

where

u_B is a function which satisfies the b.c.'s

α_k are unknown coefficients

ϕ_k are a known set of functions from a complete sequence and satisfy the b.c.'s

- Admissibility

1. The sequence must satisfy the b.c.'s

$$S(u_B) = g$$

$$S(\phi_i) = 0 \quad i = 1, N$$

Thus the b.c.'s must be satisfied independent of the parameter α_i. Also both essential and natural b.c.'s must be satisfied.

2. The sequence of trial functions must satisfy functional continuity requirements. This relates to the Sobelov Space requirements.

- Completeness of the Trial Sequence

  A sequence of linearly independent functions which satisfy admissibility requirements is complete if the (numerical) solution converges as $N \rightarrow \infty$.

  eg.: sines and cosines

  polynomials

  Bessel functions

- So far we have satisfied the d.e. on the boundary. We have a violation of the d.e. in the interior and this defines the ***Residual Error on the interior domain:***

$$\varepsilon_I = L(u_{app}) - p$$

$$\Rightarrow$$

$$\varepsilon_I = L(u_B) + \sum_{k=1}^{N} \alpha_k L(\phi_k) - p$$

For exact solutions, $\varepsilon_I = 0$ for all x in Ω. ε_I ***represent a point error measure.***

- Thus we must solve for N unknown coefficients, α_k. Therefore we must produce N constraints!

  We require the inner produc~~e~~t of ε_I and N linearly independent functions to be zero. Thus we require orthogonality between ε_I and a set of weighting functions. Therefore, we must drive the error to zero through orthogonality requirements.

  ϕ_k's are the trial functions

  w_i's are the test/weighting functions

Hence

$$\langle \varepsilon_I, w_i \rangle = \int_V \varepsilon_I w_i dx = 0 \quad \text{for} \quad i = 1, N$$

This produces the following set of linearly independent linear/nonlinear algebraic equations:

$$\sum_{k=1}^{N} \alpha_k \langle w_i, L(\phi_k) \rangle = -\langle L(u_B) - p, w_i \rangle \quad i = 1, N$$

k = column index

i = row index

thus

$$a_{ik}\alpha_k = c_i$$

where

$$a_{ik} = \langle w_i, L(\phi_k) \rangle$$
$$c_i = -\langle L(u_B) - p, w_i \rangle$$

<u>Selection of Weighting Functions</u>

1. Point Collocation

$$w_i = \delta(x - x_i)$$

- Thus we satisfy $\varepsilon_I = 0$ at selected points.
- The procedure is very simple.
- The procedure does not produce symmetrical positive definite matrices.
- For FE applications, higher order derivatives require higher order interpolating functions than of the "optimal" Galerkin method.

2. Least Squares

Minimize coefficients α w.r.t. the total square error:

$$F = \langle \varepsilon_I, \varepsilon_I \rangle$$

$$\Rightarrow$$

$$\frac{\partial F}{\partial \alpha_j} = 0 \quad j = 1, N$$

It turns out that weighting functions are:

$$w_i = L(\phi_i)$$

- The matrix produced is symmetrical and positive definite.
- The method involves a lot of computational effort.

3. Galerkin Method

$$w_i = \phi_i$$

- The method is computationally a lot simpler than least squares.
- The matrix produced depends on operator L

  L self adjoint leads to a symmetrical matrix.

  L positive definite leads to a positive definite matrix.

Note that a symmetrical positive definite matrix is a desirable property.

Problems with Weighted Residual Methods

1. We have defined functions over the entire domain. It is therefore is difficult to satisfy the b.c.'s for irregularly shaped domains (eg. 2-D).

   The solution to this problem is to define localized functions, i.e. FE's.

   However now inter-element functional continuity requirements must be met.

   We note that different functional continuity requirement exist for trial and test functions. Let's try to remedy this by lowering functional continuity requirements.

2. We have required that all b.c.'s be satisfied (e.g. 2nd order operator both function and its derivative must be satisfied at appropriate parts of the boundary).

   The solution to this problem is to back off on this requirement and satisfy only "essential" b.c.'s. This is a lot easier!

Thus we overcome inherent difficulties by using "weak" weighted residual formulations.

Properties of an Operator and Associated b.c.'s

- Properties of an operator are very similar to those of matrices.

$$\langle v, L(u) \rangle = \langle u, L^*(v) \rangle + \int_{\Gamma} \{F(v)\,G(u) - F(u)\,G^*(v)\}\, d\Gamma$$

- The $F(v)\,G(u)$ terms appear in the first half of the integration by parts procedure while the $F(u)\,G^*(v)$ term appears in the second half of the integration by parts procedure.

- If $L = L^*$, the operator is self adjoint (symmetric).

- If $\langle L(u), u \rangle > 0$ for all $u \neq 0$ which satisfy the homogeneous b.c.'s, the operator is positive-definite (examine the halfway point of the integration by parts procedure).

- The b.c.'s are defined as:

$F(u)$	essential b.c.'s
$G(u)$	natural b.c.'s

  These boundary terms fall out of the integration by parts procedure. Thus all we need do is the integration by parts and match the terms to the formula.

  e.g.: 2nd order operator: $L(u) = \dfrac{\partial^2 u}{\partial x^2}$

  u specified, essential b.c.

  $\dfrac{du}{dx}$ specified, natural b.c.

- Note that we need only go to the halfway point of the integration by parts, to find $F(v)$ $G(u)$, i.e. these terms have appeared at the halfway point (or symmetrical point).

Weak Forms

1. Fundamental Weak Form: Relax b.c. requirements and allow an error in the natural b.c.

$$L(u) - p = 0 \text{ in } \Omega$$

$$S_E(u) = \bar{g}_E \text{ on } \Gamma_E$$

$$S_N(u) = \bar{g}_N \text{ on } \Gamma_N$$

$$u_{app} = u_B + \sum_{i=1}^{N} \alpha_i \phi_i$$

Now select u_B such that $S_E(u_B) = \bar{g}_E$

and require that $S_E(\phi_i) = 0 \quad i = 1, N$

Since we're not satisfying the error in the interior or the natural boundary, we require that:

$$\langle \varepsilon_I, w_j \rangle + \langle \varepsilon_{B,N}, w_j \rangle = 0$$

where

$$\varepsilon_I = L(u_{app}) - p$$

$$\varepsilon_{B,N} = -S_N(u_{app}) + \bar{g}_N$$

This establishes the "fundamental weak form".

We have relaxed admissibility conditions such that only the essential b.c.'s must be satisfied by the trial functions. ***This is a lot easier.***

2. <u>Symmetrical Weak Form</u>

Perform an integration by parts on the $\langle \varepsilon_I, w_j \rangle$ term until "<u>symmetrical weak form</u>" is reached (derivative orders on ϕ_j and w_j match).

- Consequences

i. Unknown parts of the boundary error term drops (this happens as a result of how the natural b.c.'s were defined).

ii. Functional continuity (Sobelov Space) requirements have been lowered for the trial functions and raised for weighting (test) functions and are now the same. This is important for Galerkin methods where the trial and test functions used are from the same set of functions, ϕ_i.

Therefore we have achieved the lowest Sobelov space. This results in less work and fewer unknowns.

This is very important for FE's in order to lower inter-element functional continuity requirements.