Graduate Algebra, Fall 2014
Lecture 10

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2014-09-17

1 Group Theory

1.15 Group actions (continued)

Theorem 1 (Class equation)**.** *Let G be a finite group acting on a finite set X.*

1. *$X = \sqcup O_i$ where the O_i are the orbits of G on X.*
2. *If $x \in X$ then $|O(x)| = [G : \mathrm{Stab}_G(x)]$.*
3. *In each orbit of G acting on X choose an element x_i. Then*
$$|X| = \sum [G : \mathrm{Stab}_G(x_i)]$$
4. *In each conjugacy class in G with more than one element select an element g_i. Then*
$$|G| = |Z(G)| + \sum [G : C_G(g_i)]$$

Proof. (1): Every $x \in X$ lies in the orbit $O(x)$ so get a disjoint union.

(2): Consider the map $f : G \to O(x)$ sending g to gx. What is the preimage of $y \in O(X)$ in G? Suppose $gx = hx = y$. This is equivalent to $gh^{-1} \in \mathrm{Stab}_G(x)$ and so $|f^{-1}(y)| = |\mathrm{Stab}_G(x)|$ for all $y \in O(x)$. Thus $|G| = \sum_{y \in O(x)} |f^{-1}(y)| = |O(x)||\mathrm{Stab}_G(x)|$ and the result follows.

(3): The RHS is the sum of sizes of all orbits, which equals the size of all X as X is a disjoint union of orbits.

(4): Take the conjugacy action of G on itself, in which case stabilizers are centralizers and orbits are conjugacy classes. Thus $|G| = \sum [G : C_G(g_i)]$ if we choose g_i in all conjugacy classes. If a conjugacy class consists of the one element g_i then $g_i \in Z(G)$ and $\mathrm{Stab}_G(g_i) = G$. Thus we get the desired result. □

Remark 1. To apply the class equation for the action of G on itself it is crucial to choose representatives in each orbit, namely a set S such that each $s \in S$ is in an orbit, and every orbit contains an element of S.

When the action is conjugation of G on itself we need to find representatives in each conjugacy class.

Example 2. Some conjugacy classes.

1. The conjugacy classes in $\mathrm{GL}(2, \mathbb{R})$. The conjugacy classes in $\mathrm{GL}(2, \mathbb{C})$. (Jordan canonical forms; done last time.)
2. The conjugacy classes in S_n. Every $\sigma \in S_n$ can be written uniquely (up to permutation) as a product $\prod c_i$ of disjoint cycles. The multiset of lengths of these cycles is the **cycle type** of σ. The proposition I proved in class was that two permutations are in the same conjugacy class iff they have the same cycle type (up to permutation). The idea is that if $c = (i_j)$ is a cycle then $\tau c \tau^{-1} = (\tau(i_j))$ and so if

we write $\sigma = \prod c_i, \sigma' = \prod c_i'$ cycles of same lengths, writing the cycles one below the other gives a τ such that $\tau\sigma\tau^{-1} = \prod \tau c_i \tau^{-1} = \prod c_i' = \sigma'$.

Thus the conjugacy classes of S_n are parametrized by partitions $n = a_1 + \cdots + a_k$ with $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ and the conjugacy class corresponding to this partition consists of all products of disjoint cycles of lengths $a_1, \ldots, a_k$.

1.16 The Sylow theorems

Definition 3. 1. If G is a finite group with $p^r m$ elements, where $p \nmid m$ and $r > 0$, a p-Sylow subgroup of G is any subgroup of order p^r.

2. A p-group is any group whose cardinality is of the form p^r. We denote $\mathrm{Syl}_p(G)$ the set of all p-Sylow subgroups of G.

Example 4. 1. $\mathrm{Syl}_p(\mathbb{Z}/p^r m\mathbb{Z}) = \{m\mathbb{Z}/p^r m\mathbb{Z} \cong \mathbb{Z}/p^r\mathbb{Z}\}$.

2. $\mathrm{Syl}_2(S_3) = \{\langle(12)\rangle, \langle(13)\rangle, \langle(23)\rangle\}$ and $\mathrm{Syl}_3(S_3) = \{\langle(123)\rangle\}$.

Theorem 5. *Let G be a finite group with $p^r m$ elements, where $p \nmid m$ and $r > 0$.*

1. *G has at least one p-Sylow subgroup and denote $n_p = |\mathrm{Syl}_p(G)| \geq 1$.*

2. *$n_p \equiv 1 \pmod p$.*

3. *Every p-subgroup of G is contained in a p-Sylow subgroup.*

4. *If $P \in \mathrm{Syl}_p(G)$ then $\mathrm{Syl}_p(G) = \{gPg^{-1} | g \in G\}$, i.e., every two p-Sylow subgroups are conjugate.*

5. *$n_p \mid m$.*

Remark 2. If $n_p = 1$ then $\mathrm{Syl}_p(G) = \{P\}$ with $P \lhd G$. Indeed, $gPg^{-1} \in \mathrm{Syl}_p(G)$ and so P must be normal.

Example 6. Suppose $p > q$ are primes such that $q \nmid p - 1$. Then $|G| = pq$ implies G is cyclic.

Indeed, $n_p \mid q$ sp $n_p = 1, q$. Also $p \mid n_p - 1$ so $n_p = 1$ as $p > q - 1$. Thus $\mathrm{Syl}_p(G) = \{P\}$ with $|P| = p$ and so P cyclic of order p and normal. Similarly $n_q = 1, p$ and since $q \nmid p - 1$ it follows that $n_q = 1$ so $\mathrm{Syl}_q(G) = \{Q\}$ with $|Q| = q$ so Q cyclic of order q and normal.

Since $(p, q) = 1$ we get $P \cap Q = 1$ and $P, Q \lhd G$ so $PQ \cong P \times Q \cong \mathbb{Z}/pq\mathbb{Z}$ by the Chinese Remainder Theorem. But comparing orders get $G = PQ$ is cyclic as desired.