Graduate Algebra, Fall 2014
Lecture 12

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1 Group Theory

1.17 Application of the Sylow theorems: classifying finite groups (continued)

Example 1. (Continued from last time) If $|G| = p^2 q$ then G has a normal Sylow subgroup.

Proof. We reduced to the case $|G| = 12$. The method to tackle this case is very useful!

Assuming G has no normal Sylow subgroup implies, as before, that $n_2 = 3$ and $n_3 = 2^2 = 4$. Every element of G of order 3 generates a 3-Sylow subgroup and every two distinct 3-Sylow subgroups intersect trivially. Thus the number of elements of G of order 3 is $(3-1)n_3 = 8$. Let $P \in \mathrm{Syl}_2(G)$, of order 4. Then an element of G is either among the 8 of order 3, or among the remaining 4 which make up all of P. Thus P is the only 2-Sylow subgroup and it is normal. □

The Sylow theorems and semidirect products

In the previous example write P for a p-Sylow group and Q for a q-Sylow group. The content of the example is that one of P and Q is normal. Then $PQ = QP$ is a subgroup of G and $P \cap Q = 1$ since $(|P|, |Q|) = 1$ and $|P \cap Q| \mid |P|, |Q|$. Thus $|PQ| = |P||Q|/|P \cap Q| = p^2 q = |G|$ so $G = PQ$. If $P \lhd G$ then $G \cong P \rtimes Q$ and if $Q \lhd G$ then $G \cong Q \rtimes P$.

This procedure is very effective: use the Sylow theorems (and counts of elements of a certain order) to produce normal Sylow subgroups and the use other Sylow subgroups to write the group as a semidirect product.

Two questions arise about semidirect products:

First, given $G = N \rtimes_\phi H$ for a homomorphism $\phi : H \to \mathrm{Aut}(N)$ and isomorphisms $N \cong N'$ and $H \cong H'$ explicitly write a homomorphism $\phi' : H' \to \mathrm{Aut}(N')$ such that $G \cong N' \rtimes_{\phi'} H'$. This is possible because of the isomorphisms. In particular, if $N \lhd G$ and H is a subgroup of G such that $G = NH$ and $N \cap H = 1$ then we know that $G \cong N \rtimes H$ given by $H \to \mathrm{Inn}(N) \subset \mathrm{Aut}(N)$ sending $h \in H$ to the inner automorphism $n \mapsto hnh^{-1}$. Often we'd like to describe this explicit semidirect product given by conjugation in terms of simpler isomorphic groups.

The example from the homework is relevant. We saw that $G = \{\begin{pmatrix} * & * \\ & 1 \end{pmatrix}\}$ contained a normal $N = \{\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}\}$ and the diagonal subgroup H with the property that $G = NH$ and $N \cap H = 1$. The map $d : x \mapsto \begin{pmatrix} x & \\ & 1 \end{pmatrix}$ given an isomorphism $(\mathbb{Z}/p\mathbb{Z})^\times \cong H$ and the map $u : x \mapsto \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}$ gives an isomorphism $\mathbb{Z}/p\mathbb{Z} \cong N$. We would like $\phi : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathrm{Aut}(\mathbb{Z}/p\mathbb{Z})$ such that $G \cong \mathbb{Z}/p\mathbb{Z} \rtimes (\mathbb{Z}/p\mathbb{Z})^\times$. In other words, we would like a homomorphism ϕ such that $u(\phi(h)n) = d(h)u(n)d(h)^{-1}$ since the original homomorphism is conjugation.

But $d(h)u(n)d(h)^{-1} = u(hn)$ so $\phi(h)n = hn$ sends h to the multiplication by h automorphism of $\mathbb{Z}/p\mathbb{Z}$.

Second, given N and H we would like to tell whether two semidirect products $N \rtimes H$ are isomorphic. The simplest example is the following: if $(|H|, |\operatorname{Aut}(N)|) = 1$ then $N \rtimes H \cong N \times H$. Indeed, if $\phi : H \to \operatorname{Aut}(N)$ then $\operatorname{Im}(\phi)$ has order dividing both $|H|$ and $|\operatorname{Aut}(N)|$ and so $\phi = 1$ giving the direct product.

What is $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$? They are parametrized by homomorphisms $\phi : \mathbb{Z}/2\mathbb{Z} \to \operatorname{Aut}(\mathbb{Z}/3\mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z}$ and such ϕ are determined by $\phi(1)$ which is either the identity map or the multiplication by 2 map. Thus there are at most two semidirect products. We already know that the direct product $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ corresponds to $\phi(1) = \mathrm{id}$. But $S_n \cong A_n \rtimes \langle (12) \rangle$ and so $S_3 \cong \mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ so we constructed two nonisomorphic examples and thus these are all the semidirect products.

In general listing or even counting the isomorphism classes of semidirect products is hard.

Theorem 2. *Suppose $H = \langle a \rangle$ is finite cyclic and N is a group. Suppose $f, g : H \to \operatorname{Aut}(N)$ are homomorphisms such that $\operatorname{Im} f$ and $\operatorname{Im} g$ are conjugate subgroups of $\operatorname{Aut}(N)$. Then $N \rtimes_f H \cong N \rtimes_g H$.*

Before proving this theorem let's see some applications.

Example 3.

1. What are the possibilities for $S_3 \rtimes \mathbb{Z}/2\mathbb{Z}$? A homomorphism $f : \mathbb{Z}/2\mathbb{Z} \to \operatorname{Aut}(S_3) = \operatorname{Inn}(S_3) \cong S_3$ sends 0 to the identity permutation and 1 to $f(1) \in S_3$ which gives the associated inner automorphism of S_3. Since f is a homomorphism, $1 = f(2) = f(1)^2$ and so $f(1) \in \{1, (12), (13), (23)\}$. Thus there are four choices for the homomorphism f: f_τ sending 1 to τ for $\tau \in \{1, (12), (13), (23)\}$. By inspection $\operatorname{Im} f_\tau = \langle \tau \rangle$.

   We know the conjugacy classes of S_n: they are indexed by cycle types. Thus every two transpositions are conjugate and so $\operatorname{Im} f_\tau$ and $\operatorname{Im} f_{\tau'}$ if τ and τ' are transpositions. Thus we get two nonisomorphic semidirect products: $S_3 \times \mathbb{Z}/2\mathbb{Z}$ corresponding to f_1 and $S_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ corresponding to f_τ for τ any of the three transpositions.

2. What are the nonisomorphic semidirect products $(\mathbb{Z}/17\mathbb{Z})^2 \rtimes \mathbb{Z}/3\mathbb{Z}$? The semidirect products are indexed by homomorphisms $f : \mathbb{Z}/3\mathbb{Z} \to \mathrm{GL}(2, \mathbb{F}_{17})$ and we would like to classify there homomorphisms up to conjugate image in $\mathrm{GL}(2, \mathbb{F}_{17})$. Certainly the trivial map gives the direct product. What about the nontrivial map? Then $g = f(1)$ is a matrix in $\mathrm{GL}(2, \mathbb{F}_{17})$ such that $g \neq I_2$ and $g^3 = I_2$. The Jordan canonical form for matrices over $\mathbb{F}_{17}$ then tells us that g is conjugate to $\begin{pmatrix} -1 & \\ -1 & 1 \end{pmatrix}$. Thus there exist exactly two nonisomorphic semidirect products.