Graduate Algebra, Fall 2014
Lecture 13

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1 Group Theory

Sylow theorems and semidirect products (continued)

Theorem 1. *Suppose $H = \langle a \rangle$ is finite cyclic of prime power order and N is a group. Suppose $f, g : H \to \mathrm{Aut}(N)$ are homomorphisms such that $\mathrm{Im}\, f$ and $\mathrm{Im}\, g$ are conjugate subgroups of $\mathrm{Aut}(N)$. Then $N \rtimes_f H \cong N \rtimes_g H$.*

Proof. Since $\mathrm{Im}\, f$ and $\mathrm{Im}\, g$ are conjugate there exists $\sigma \in \mathrm{Aut}(N)$ such that $\sigma \,\mathrm{Im}\, f \sigma^{-1} = \mathrm{Im}\, g$ and so $\sigma\langle f(a)\rangle = \langle \sigma f(a)\rangle = \langle g(a)\rangle$. Thus $g(a)^k = \sigma f(a)\sigma^{-1}$ for some integer k coprime to $\mathrm{ord}(g(a)) = \mathrm{ord}(f(a))$. Since $f(a)$ and $g(a)$ are homomorphisms we deduce that for $h \in \langle a \rangle = H$ we have $\sigma f(h)\sigma^{-1} = g(h)^k = g(h^k)$, or $\sigma f(h) = g(h^k)\sigma$.

Consider the map $N \rtimes_f H \to N \rtimes_g H$ defined as $\phi(n, h) = (\sigma(n), h^k)$. Let's check it's a homomorphism.

$$\begin{aligned}
\phi((n,h) \cdot_f (n', h')) &= \phi((nf(h)(n'), hh')) \\
&= (\sigma(nf(h)(n')), (hh')^k) \\
&= (\sigma(n)\sigma f(h)(n'), h^k (h')^k) \\
&= (\sigma(n) g(h^k)\sigma(n'), h^k (h')^k) \\
&= (\sigma(n), h^k) \cdot_g (\sigma(n'), (h')^k) \\
&= \phi(n,h) \cdot_g \phi(n', h')
\end{aligned}$$

since H is abelian. □

1.17 Application of the Sylow theorems: classifying finite groups (continued)

Example 2. If $|G| = 12$ then either G has a normal 3-Sylow subgroup or it is $\cong A_4$ and has a normal 2-Sylow subgroup.

Proof. From the previous example, there is a unique normal 2-Sylow subgroup P of G. Consider the conjugation action of G on the set of subgroups of G. Then 3-Sylow subgroups form a conjugacy class and if $Q \in \mathrm{Syl}_3(G)$ then $n_3 = 4 = [G : \mathrm{Stab}_G(Q)]$. Certainly $Q \subset \mathrm{Stab}_G(Q)$ and $[G : Q] = 4$ so we deduce that $\mathrm{Stab}_G(Q) = Q$.

Aside: What is $\mathrm{Stab}_G(Q)$? It is the set of $g \in G$ such that $gQg^{-1} = Q$, i.e., the normalizer of Q in G, which is the largest subgroup of G in which Q is normal.

Consider the conjugation action of G on $\mathrm{Syl}_3(G)$, which has 4 elements. This gives a homomorphism $f : G \to S_4$. If $g \in \ker f$ then $gQg^{-1} = Q$ for each $Q \in \mathrm{Syl}_3(G)$ and we already saw that $g \in Q$ so $\ker f \subset Q$. But $Q \cong \mathbb{Z}/3\mathbb{Z}$ so either $\ker f = 1$ or $\ker f = Q$. The latter cannot be since Q is not normal in G but $\ker f$ is. Thus f is injective $G \hookrightarrow S_4$. The only subgroup of S_4 with 12 elements is A_4 so $G \cong A_4$, which has $(\mathbb{Z}/2\mathbb{Z})^2$ as a normal subgroup as the subgroup of products of disjoint transpositions. □

Example 3. Suppose $|G| = 30$.

1. Then G has a normal 3-Sylow and a normal 5-Sylow subgroups.
2. There are four isomorphism classes of groups of order 30: $\mathbb{Z}/30\mathbb{Z}$, D_{30}, $\mathbb{Z}/5\mathbb{Z} \times S_3$ and $\mathbb{Z}/3\mathbb{Z} \times D_{10}$.