Graduate Algebra, Fall 2014
Lecture 14

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1 Group Theory

1.17 Application of the Sylow theorems: classifying finite groups (continued)

Proposition 1. *Suppose $|G| = 30$.*

1. *Then G has a normal 3-Sylow and a normal 5-Sylow subgroups.*
2. *There are four isomorphism classes of groups of order 30: $\mathbb{Z}/30\mathbb{Z}$, D_{30}, $\mathbb{Z}/5\mathbb{Z} \times S_3$ and $\mathbb{Z}/3\mathbb{Z} \times D_{10}$.*

Proof. (1): Let $P \in \mathrm{Syl}_5(G)$ and $Q \in \mathrm{Syl}_3(G)$. If P or Q is normal then PQ is a group and is a semidirect product of P and Q. As $(3, \varphi(5)) = (5, \varphi(3)) = 1$ it follows that the semidirect product is a direct product so $PQ \cong \mathbb{Z}/15\mathbb{Z}$ is a subgroup of G. Since PQ has index 2 in G it is normal in G and we deduce that both P and Q are normal in G: indeed, $gPQg^{-1} = gP \times Qg^{-1} = gPg^{-1} \times gQg^{-1} = P \times gQg^{-1} = P \times Q$.

Suppose now that neither P nor Q is normal in G. Then $n_5 = 6$ and $n_3 = 10$. The number of elements of G of order 5 is $(5-1)n_5 = 24$ and the number of elements of order 3 are similarly $(3-1)n_3 = 20$ altogether giving more than the 30 elements of G. Thus P and Q are both normal in G.

(2): If G is abelian then the structure theorem implies that $G \cong \mathbb{Z}/30\mathbb{Z}$. Suppose G is not abelian. Then G has $\mathbb{Z}/15\mathbb{Z}$ as a normal subgroup. If $H \in \mathrm{Syl}_2(G)$ then $H \cap PQ = 1$ and certainly $PQH = G$ so $G \cong PQ \rtimes H$ with morphism $\mathbb{Z}/2\mathbb{Z} \to (\mathbb{Z}/15\mathbb{Z})^\times$. This means $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and there are exactly 4 such homomorphisms: the trivial one, giving $\mathbb{Z}/30\mathbb{Z}$, $1 \mapsto (1,0)$, $1 \mapsto (0,2)$ and $1 \mapsto (1,2)$. Here the list of 4 is complete. $\square$

Proposition 2. *Let G be a finite group of order 60. The following are equivalent.*

1. *$n_5 > 1$, i.e., G has more than one 5-Sylow subgroup.*
2. *G is simple.*
3. *$G \cong A_5$.*

Proof. $3 \Rightarrow 1$: $\mathrm{Syl}_5(A_5)$ contains $\langle (12345) \rangle$ and $\langle (13245) \rangle$ thus $n_5 > 1$.

$1 \Rightarrow 2$: Suppose that $H \lhd G$ and $n_5 > 1$ in which case we know $n_5 \mid 12$ with $n_5 \equiv 1 \pmod 5$ so $n_5 = 6$. Let $P \in \mathrm{Syl}_5(G)$ and so $[G : N_G(P)] = n_5 = 6$ giving $|N_G(P)| = 10$.

Case A: If $5 \mid |H|$ then $P \subset H$ and so all 5-Sylow are in H (by normality of H and the fact that all 5-Sylow subgroups are conjugate) giving $|H| \geq 6 \cdot (5-1) + 1 = 25$ (any two 5-Sylow subgroups have size 5 so are either the same or intersect only in 1) and since $|H| \mid 60$ we deduce $|H| = 30$. Now $\mathrm{Syl}_5(H)$ is unique by the previous proposition and $\mathrm{Syl}_5(G) = \mathrm{Syl}_5(H)$ contradicting $n_5 > 1$.

Case B: If $5 \nmid |H|$ and H is proper the only possibilities left are $|H| \mid 12$.

Case B1: If $|H| = 2, 3, 4$ then G/H has size $15, 20, 30$. If 30 then $\mathrm{Syl}_5(G/H)$ is normal in G/H by the previous proposition; if $20 = 2^2 \cdot 5$ then again $\mathrm{Syl}_5(G/H)$ is normal in G/H by the general p^2q example. If $15 = 3 \cdot 5$ then G/H is $G/H \cong \mathbb{Z}/15\mathbb{Z}$ and again $\mathrm{Syl}_5(G/H)$ is normal in G/H. Let H' be the preimage of

$\mathrm{Syl}_5(G/H)$ in which case $H' \lhd G$ (showed this in class). But then $5 \mid |H'|$ and we proceed as above to get a contradiction.

Case B2 and $2 \Rightarrow 3$ next time. $\square$