Graduate Algebra, Fall 2014
Lecture 15

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1 Group Theory

1.17 Application of the Sylow theorems: classifying finite groups (continued)

We used the following lemma several times.

Lemma 1. *Suppose H is a subgroup of the finite group G and $p \nmid |G|/|H|$. Then $\mathrm{Syl}_p(H) \subset \mathrm{Syl}_p(G)$. If H is normal then in fact $\mathrm{Syl}_p(H) = \mathrm{Syl}_p(G)$.*

Proof. If $p \nmid |G|$ then both sets are empty. If $p \mid |G|$ then every p-Sylow of H is contained in some p-Sylow of G and by counting orders, every p-Sylow of H gives a p-Sylow of G.

Suppose H is normal and $P \in \mathrm{Syl}_p(H) \subset \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_p(G)$. Then $Q = gPg^{-1}$ for some $g \in G$ and thus $Q \subset gHg^{-1} = H$. □

Lemma 2. *Let G be a simple group and H a subgroup.*

1. *Then $[G : H]! \geq |H|$.*
2. *$n_p! \geq |G|$ for all $p \mid |G|$.*
3. *If $|G| = 60$ and H is a subgroup then $[G : H] \geq 5$; if $[G : H] = 5$ then $G \cong A_5$. If $n_2 = 5$ then $G \cong A_5$.*

Proof. (1): Note that G acting on G/H by left multiplication is transitive and gives a nontrivial homomorphism $f : G \to S_{G/H}$. The kernel $\ker f \lhd G$ and since G is simple we deduce that $\ker f = 1$ and so $G \hookrightarrow S_{[G:H]}$. This gives the first part.

(2): If $P \in \mathrm{Syl}_p(G)$ then $[G : \mathrm{Stab}_G(P)] = n_p$ and so (2) follows from (1).

(3): The inequality follows from $4! < 60 < 5!$. If $[G : H] = 5$ then (1) gives $G \hookrightarrow S_5$. If $G \neq A_5$ then by $S_5/A_5 \cong \mathbb{Z}/2\mathbb{Z}$ we deduce that $S_5 = A_5G$ and so that $|A_5 \cap G| = |A_5||G|/|A_5G|$. But $A_5 \cap G$ is then a normal subgroup of G. Thus $G \cong A_5$. □

Proposition 3. *Let G be a finite group of order 60. The following are equivalent.*

1. *$n_5 > 1$, i.e., G has more than one 5-Sylow subgroup.*
2. *G is simple.*
3. *$G \cong A_5$.*

Proof. **Case B2:** If $|H| = 6, 12$ then $\mathrm{Syl}_3(H) = \mathrm{Syl}_3(G)$. Either $\mathrm{Syl}_3(H)$ is then a normal subgroup of G and so we take $H = \mathrm{Syl}_3(G)$ with 3 elements or $H \cong A_4$ in which case $\mathrm{Syl}_2(H) = \mathrm{Syl}_2(G) = (\mathbb{Z}/2\mathbb{Z})^2$ which is normal in H and thus in G. Again replace H by $\mathrm{Syl}_2(H)$ of size 4.

$2 \Rightarrow 3$:

Suppose $P \in \mathrm{Syl}_2(G)$. Then $n_2 > 1$ by simplicity so $n_2 \in \{3, 5, 15\}$ and by the previous lemma $n_2 \geq 5$ and if $n_2 = 5$ then $G \cong A_5$.

Suppose now that $n_2 = 15$. Suppose $P \neq Q$ are in $\mathrm{Syl}_2(G)$. Then P, Q have order 4 and so are abelian, either $\cong (\mathbb{Z}/2\mathbb{Z})^2$ or $\mathbb{Z}/4\mathbb{Z}$. If $P \cap Q = 1$ for all $P \neq Q$ then the number of elements of order 2 or 4 is $(4-1)n_2 = 45$. Also $n_5 = 6$ so the number of elements of order 5 is 24 and in total we have too many elements. Thus $P \cap Q = \mathbb{Z}/2\mathbb{Z}$ for some $P, Q \in \mathrm{Syl}_2(G)$.

Let $M = N_G(P \cap Q)$. Since P and Q are abelian, $P, Q \subset M$ and so $|M| > 4$ and by definition of normalizer $P \cap Q$ is normal in M. Since G is simple, $M \neq G$ and so $|M| \leq 30$. So $4 \mid |M| \mid 60$ and $|M| \leq 30$ so $|M| = 12, 20$. The case $|M| = 20$ fails because then $[G : M] = 3 < 5$ and the case $|M| = 12$ gives $[G : M] = 5$ and so $G \cong A_5$ from the lemma.

$\square$