Graduate Algebra, Fall 2014
Lecture 17

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1 Group Theory

1.19 Duals (continued)

Proposition 1. *Let G be a group.*

1. *$[G,G] \lhd G$ and G^{ab} is an abelian group.*
2. *If $f : G \to A$ is a homomorphism to an abelian group then there exists a homomorphism $\overline{f} : G^{\mathrm{ab}} \to A$ such that $G \to G^{\mathrm{ab}} \to A$ commutes.*
3. *$\mathrm{Hom}(G, A) \cong \mathrm{Hom}(G^{\mathrm{ab}}, A)$.*
4. *If G is finite then $(G^\vee)^\vee \cong G^{\mathrm{ab}}$.*

Proof. The first part follows from $x[a,b]x^{-1} = [xax^{-1}, xbx^{-1}]$.

Suppose $f : G \to A$ is a homomorphism. Then $[G,G] \subset \ker f$ and by the first isomorphism theorem we deduce the second part.

Third part: The map $f \mapsto \overline{f}$ gives the isomorphism.

Fourth part: By part (3) it suffices to show the statement for G abelian, in this case finite. Consider $G \to (G^\vee)^\vee$ sending g to $\phi_g : f \mapsto f(g)$ for $f \in G^\vee$. This is an injective homomorphism. Indeed, if $\phi_g(f) = 0$ then $f(g) = 0$ for all f. If $g \neq 1$ then consider the natural projection $G \to \langle g \rangle$ and send g to any nonzero element to get $f : G \to A$ such that $f(g) \neq 0$, getting a contradiction. □

Example 2.

1. From the homework $[S_n, S_n] = [A_n, A_n] = A_n$ so $S_n^{\mathrm{ab}} \cong \mathbb{Z}/2\mathbb{Z}$ and $A_n^{\mathrm{ab}} = 1$.
2. $[D_{2n}, D_{2n}] = \langle R^2 \rangle$.
3. It is also the case that $\mathrm{GL}(n, \mathbb{F}_q)^{\mathrm{ab}} \cong \mathbb{F}_q^\times$ when $(n,q) \neq (2,3)$.

1.20 Solvable groups and nilpotent groups

Definition 3. A finite group G is said to be **solvable** if there exist subgroups $G = G_0 \rhd G_1 \rhd \ldots \rhd G_s = 1$ such that G_{i+1} is normal in G_i and such that G_i/G_{i+1} is abelian.

Remark 1. One can show that if G is solvable then the quotients above can be chosen to be cyclic of prime order.

Proposition 4.

1. *If H is a normal subgroup of the finite group G such that G/H is abelian then $[G,G] \subset H$.*
2. *A finite group G is solvable iff the sequence of subgroups $G^0 = G$, $G^{i+1} = [G^i, G^i]$ terminates in $G^m = 1$ for some m.*

Proof. (1): $G \to G/H$ factors through $G \to G^{\mathrm{ab}} = G/[G,G] \to G/H$ so $[G,G]$ is in the kernel of the projection map $G \to G/H$ so $[G,G] \subset H$.

(2): Next time. □

Example 5.

1. S_3 is solvable taking $S_3 \rhd A_3$.
2. S_4 is solvable taking $S_4 \rhd A_4 \rhd (\mathbb{Z}/2\mathbb{Z})^2 \rhd 1$.
3. The group B of upper triangular matrices in $\mathrm{GL}(n, R)$ is solvable, taking

$$\left\{\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}\right\} \rhd \left\{\begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}\right\} \rhd \left\{\begin{pmatrix} 1 & 0 & * & * \\ & 1 & 0 & * \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}\right\} \rhd \ldots \rhd \left\{\begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right\}$$

4. The group A_5 is not solvable (and this has deep consequences in number theory). Indeed, $G^0 = A_4$, $G^1 = [A_5, A_5] = A_5$ and so $G^n = A_5$ always.