Graduate Algebra, Fall 2014
Lecture 18

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1 Group Theory

1.20 Solvable groups and nilpotent groups (continued)

Proposition 1. *1. If H is a normal subgroup of the finite group G such that G/H is abelian then $[G,G] \subset H$.*

2. A finite group G is solvable iff the sequence of subgroups $G^0 = G$, $G^{i+1} = [G^i, G^i]$ terminates in $G^m = 1$ for some m.

Proof. (1): Last time.

(2): If $G = G_0 \rhd G_1 \rhd G_2 \rhd \ldots \rhd G_s = 1$ then (1) shows that $G^1 = [G,G] \subset G_1$ so $G^2 = [G^1, G^1] \subset [G_1, G_1] \subset G_2$ and so on we deduce that $G^\bullet$ terminates. Reciprocally, we can take $G_i = G^i$ as $G_{i+1} \lhd G_i$ and the quotient is G_i^{ab}. $\square$

Definition 2. A group G is **nilpotent** if the sequence $G_0 = G$, $G_{i+1} = [G_i, G]$ terminates in $G_k = 1$ for some k.

Proposition 3. *Every abelian group is nilpotent. Every nilpotent group is solvable.*

Example 4. 1. Upper triangular matrices with 1-s on the diagonal are nilpotent.

2. Upper triangular matrices are solvable but not nilpotent.

3. D_8 and Q are nilpotent.

4. The group S_3 is solvable but not nilpotent.

Proposition 5. *Let G be a nilpotent group. Then every Sylow subgroup of G is normal and so G is a direct product of its Sylow subgroups.*

Definition 6. A **composition series** for a group G is a sequence $G = G_0 \rhd G_1 \rhd \ldots G_k = 1$ such that G_i/G_{i+1} is simple.

Theorem 7 (Jordan-Hölder)**.** *Let G be a nontrivial finite group.*

1. Then G has a composition series.

2. Every two composition series have the same length and the set simple groups arising as quotients is the same.

Example 8. 1. Suppose G has composition series G_i and H has composition series H_j. Then $G \times H$ has composition series $G_i \times H, 1 \times H_j$.

2. The composition series for $\mathbb{Z}/p^n\mathbb{Z}$ is $\mathbb{Z}/p^n\mathbb{Z} \rhd \mathbb{Z}/p^{n-1}\mathbb{Z} \rhd \ldots \rhd \mathbb{Z}/p\mathbb{Z} \rhd 1$.

3. The group S_n has composition series $S_n \rhd A_n \rhd 1$ for $n \neq 4$. The group S_4 has composition series $S_4 \rhd A_4 \rhd (\mathbb{Z}/2\mathbb{Z})^2 \rhd \mathbb{Z}/2\mathbb{Z} \rhd 1$.

1.21 Limits

We study two important constructions.

Definition 9. A **directed set** is a partially ordered set I such that if $u, v \in I$ then there exists $w \in I$ such that $u, v \leq w$.

Example 10. The following are directed sets.

1. Any totally ordered set.
2. The set of open subsets in a topological space with partial ordering given by $U \leq V$ is $U \subset V$.
3. Let X be a topological space and $x \in X$. Let $\mathcal{I} = \{U \text{ open } \subset X | x \in U\}$ partially ordered by $U \leq V$ iff $V \subset U$.

Taylor expansions and limits of groups

The prototypical setup where the concept of a direct limit of groups is essential is the algebraic formulation of Taylor series.

For an open set $I \subset \mathbb{R}$ let $C^\infty(I)$ be the set of smooth functions $f : I \to \mathbb{R}$. This becomes a group with respect to addition. If $J \subset I$ then $\mathrm{res}_{I/J} : C^\infty(I) \to C^\infty(J)$ sending the smooth function f with domain I to the same function f but with domain J is a group homomorphism.

What is a Taylor expansion of $f \in C^\infty(\mathbb{R})$ around $x = a$? It is a power series $T_f(x-a) = \sum_{n \geq 0} c_n(x-a)^n$ which converges absolutely on some small neighborhood I of a. There are two features of Taylor expansions that make them interesting algebraically:

1. A Taylor expansion $T_f(x-a)$ can make sense on a very small neighborhood of a as a function in $C^\infty(I)$ for I an open interval containing a.
2. Two functions $f \neq g \in C^\infty(\mathbb{R})$ could have the same Taylor expansion around $x = a$ as long as f and g agreen on a small neighborhood around a. For example $f(x) = e^{-x^{-2}}$ for $x \neq 0$ and $f(0) = 0$ is smooth, and so is $g(x) = f(x)$ for $x \geq 0$ and $g(x) = 0$ for $x < 0$. They are clearly different functions but have the same Taylor series around $x = a > 0$.

To make things precise algebraically, $T_f(x-a) = T_g(x-a)$ if and only if $\mathrm{res}_{\mathbb{R}/I}(f) = \mathrm{res}_{\mathbb{R}/I}(g)$ for some small enough open interval I containing a and every $T_f(x-a)$ is in some $C^\infty(I)$ for some small enough I containing a.

To make sense of Taylor series algebraically we would define the set of Taylor series of smooth functions around $x = a$ as some sort of limit $\lim C^\infty(I)$ where I ranges over small open neighborhoods containing a. In this limit two smooth functions become equal as long as they agree on some small enough I and an element of this limit is some smooth function on some small enough I.

This is the template that we'll use to make sense of direct limits in the next section, and it is at the basis of every modern geometric theory.

1.21.1 Direct limits

Definition 11. Let I be a directed set. A **direct system** of groups is a collection $\{G_u\}_{u \in I}$ together with homomorphisms $\iota_{uv} : G_u \to G_v$ for $u \leq v$, such that

1. $\iota_{uu} = \mathrm{id}$
2. If $u < v < w$ then $\iota_{uw} = \iota_{vw} \circ \iota_{uv}$

Example 12. 1. $I = \mathbb{Z}$ as a totally ordered set, $G_n = G$ a group and $\iota_{m,n} = \mathrm{id}$.

2. Let $I = \mathbb{Z}_{\geq 1}$ with the partial ordering given by $n \leq m$ if $n \mid m$. Then I is a directed set. Consider $G_n = \mathbb{Z}$ with $\iota_{n,m} : \mathbb{Z} \to \mathbb{Z}$ for $n \mid m$ given by $\iota_{n,m}(x) = xm/n$. Then this is a direct system of groups.