Graduate Algebra, Fall 2014
Lecture 19

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1 Group Theory

1.21 Limits (continued)

1.21.1 Direct limits (continued)

Proposition 1. *Suppose $(G_u)_{u\in I}$ is a direct system of groups. The* **direct limit** $G = \varinjlim G_u$ *is defined as*

$$G = \sqcup_{u\in I} G_u / \sim$$

where the equivalence relation is as follows: $g_u \in G_u$ and $g_v \in G_v$ are equivalent $g_u \sim g_v$ if $f_{uw}(g_u) = f_{vw}(g_v)$ for some $w > u, v$. Then G is a group and there exist homomorphisms $\iota_u : G_u \to G$ such that for $u < v$ have $\iota_v = \iota_{uv} \circ \iota_u$.

Remark 1. What are elements of this group? They are elements of G_u for some u large enough, and two elements in different groups are equal in the direct limit iff their image in some G_u for large enough u are equal. These are precisely the two conditions that we identified for Taylor series and direct limits provide analogues of Taylor expansions in situations where they do not appear naturally.

Proof of Proposition. Suppose $g_u \in G_u \subset G$ and $g_v \in G_v \subset G$. Pick $w \geq u, v$. Then $g_u = \iota_{uw}(g_u)$ and $g_v = \iota_{vw}(g_v)$ and we define $g_u \cdot_G g_v := \iota_{uw}(g_u) \cdot_{G_w} \iota_{vw}(g_v)$. Since the ι_{uv} maps satisfy a transitivity relation this definition makes sense.

Also define $\iota_u : G_u \to G$ as the natural inclusion, which is then a homomorphism and naturally satisfies the desired property. $\square$

Remark 2. As mentioned before, if X is a topological space and $x \in X$ then $\mathcal{I}_x = \{U \text{ open } \subset X | x \in U\}$ is a directed set. Let $C(U)$ be the abelian group (under addition) of continuous maps $f : U \to \mathbb{R}$ and for $U \supset V$ let $\mathrm{res}_{U/V} : C(U) \to C(V)$ be the restriction homomorphism. The direct limit $\varinjlim_{U\in\mathcal{I}_x} C(U)$ is called the **stalk** of continuous functions at x and its elements are called **germs** of continuous functions. From the construction germs of continuous functions are equivalence classes of continuous functions where two continuous functions are said to be equivalent if they agree on some small open neighbordhood of x.

Example 2. Consider $p^{-n}\mathbb{Z} \subset p^{-m}\mathbb{Z}$ for $n \leq m$. This forms a direct system and I claim that $\varinjlim p^{-n}\mathbb{Z} \cong \mathbb{Q}_{(p^\infty)} = \{\frac{m}{p^n} | m \in \mathbb{Z}\}$.

Consider the map $\mathbb{Q}_{(p^\infty)} \to \varinjlim p^{-n}\mathbb{Z}$ sending $\frac{m}{p^n}$ to $\frac{m}{p^n} \in p^{-n}\mathbb{Z}$. Let's how this is a well-defined homomorphism. To check that it is well-defined need to show that $\frac{p}{p^n} = \frac{mp^d}{p^{n+d}}$ are equal in $\varinjlim p^{-n}\mathbb{Z}$, i.e., need to check that $\iota_{n,n+d}(\frac{m}{p^n}) = \frac{mp^d}{p^{n+d}}$ which is clear since all ι maps are simply inclusions of one group into another.

To check that this map is a homomorphism note that $\frac{a}{p^m} + \frac{b}{p^n} = \frac{ap^{n-m}+b}{p^n}$ gets sent to itself as an element of $p^{-n}\mathbb{Z}$. But we already checked that $\frac{a}{p^m}$ and $\frac{b}{p^n}$ get sent to $\frac{ap^{n-m}}{p^n}$ and $\frac{b}{p^n}$ as elements of $p^{-n}\mathbb{Z} \subset \varinjlim p^{-n}\mathbb{Z}$. The map is now a homomorphism because addition in $p^{-n}\mathbb{Z}$ is usual addition of fractions.

Example 3.

1. If G is a group get the direct system $G \to G \to \dots$ given by the identity map. Then $\varinjlim G \cong G$.
2. Consider $\mathbb{Z}/n\mathbb{Z} \cong \frac{1}{n}\mathbb{Z}/\mathbb{Z}$. For $n \mid m$ get an inclusion $\frac{1}{n}\mathbb{Z}/\mathbb{Z} \hookrightarrow \frac{1}{m}\mathbb{Z}/\mathbb{Z}$ and this is a direct system with limit $\varinjlim \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}$.
3. In the example above $\varinjlim \mathbb{Z}/p^n\mathbb{Z} = (\mathbb{Q}/\mathbb{Z})[p^\infty]$.
4. I've already explained that germs of smooth functions at $x = a$ are the same thing as Taylor series around $x = a$.
5. Send $S_n \hookrightarrow S_{n+1}$ by permuting the first n elements. Then S_∞ can be defined as $\varinjlim S_n$.
6. Send $\mathrm{GL}(n) \hookrightarrow \mathrm{GL}(n+1)$ in block diagonal form. Then $\mathrm{GL}(\infty)$ can be thought of as $\varinjlim \mathrm{GL}(n)$.

Theorem 4 (Direct limits = categorical colimits)**.** *The* **direct limit** $\varinjlim G_u$ *of the directed system* (G_u) *satisfies the following universal property: for any group* H *and homomorphisms* $\phi_u : G_u \to H$ *commuting with the* ι_{uv}, *there exists a homomorphism* $\phi : \varinjlim G_u \to H$ *such that* $\phi_u = \phi \circ \iota_u$.

The direct limit of a directed system of groups is uniquely defined by this universal property.

Proof. Next time. □