Graduate Algebra, Fall 2014
Lecture 20

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1 Group Theory

1.21 Limits (continued)

1.21.1 Direct limits (continued)

Theorem 1 (Direct limits = categorical colimits)**.** *The* **direct limit** $\varinjlim G_u$ *of the directed system* (G_u) *satisfies the following universal property: for any group* H *and homomorphisms* $\phi_u : G_u \to H$ *commuting with the* ι_{uv}*, there exists a homomorphism* $\phi : \varinjlim G_u \to H$ *such that* $\phi_u = \phi \circ \iota_u$ *where* $\iota_u : G_u \to \varinjlim G_u$ *is the natural inclusion homomorphism.*

The direct limit of a directed system of groups is uniquely defined by this universal property.

Proof. Define $\phi : G \to H$ as follows: pick $g_u \in G_u \subset G$ and let $\phi(g_u) := \phi_u(g_u)$. Since $g_u = \iota_{uv}(g_u)$ this definition makes sense because $\phi_u(g_u) = \phi_v(\iota_{uv}(g_u))$ and the map ϕ is a homomorphism. □

Example 2. From last time $\varinjlim p^{-n}\mathbb{Z} \cong \mathbb{Q}_{(p^\infty)}$. Consider $\phi_n : p^{-n}\mathbb{Z} \to \mathbb{Q}$ sending ap^{-n} to $2ap^{-n}$. These homomorphisms commute with all $\iota_{m,n}$ and so they come from some homomorphism $\mathbb{Q}_{(p^\infty)} \to \mathbb{Q}$. Indeed this homomorphism is simple multiplication by 2.

1.21.2 Inverse limits

Definition 3. Let I be a directed set. An **inverse system** of groups is a collection $\{G_u\}_{u\in I}$ together with homomorphisms $\pi_{vu} : G_v \to G_u$ for $u \le v$, such that

1. $\pi_{uu} = \mathrm{id}$
2. If $u < v < w$ then $\pi_{wu} = \pi_{vu} \circ \pi_{wv}$

Example 4.
1. $I = \mathbb{Z}_{\ge 0}$ with usual order. $G_n = \mathbb{C}[X]_n$ are polynomials in degree $\le n$. The maps $\pi_{m,n}$ from degree m to degree $n \le m$ simply truncates polynomials. This is an inverse system.
2. $I = \mathbb{Z}_{\ge 1}$ with $m \preceq n$ iff $m \mid n$. Then $G_m = \mathbb{Z}/m\mathbb{Z}$ with $\pi_{m,n}(x) = x \mod n$ when $n \preceq m$ gives an inverse system.

Theorem 5 (Inverse limits = categorical limits)**.** *An* **inverse limit** *of the inverse system* G_u *is a group* $\varprojlim G_u$ *together with homomorphisms* $\pi_u : \varprojlim G_u \to G_u$*, with the following universal property: for any group* H *and homomorphisms* $\phi_u : H \to G_u$ *commuting with the* π_{vu}*, there exists a homomorphism* $\phi : H \to \varinjlim G_u$ *such that* $\phi_u = \pi_u \circ \phi$*.*

Inverse limits of groups exist and are unique.

Proof. Let $G = \{(g_i) | g_i \in G_i, f_{ji}(g_j) = g_i, \forall i < j\}$. This is a group. How to construct ϕ? Let $\phi(h) = (\phi_u(h)) \in \prod G_u$. It's easy to check that in fact $\phi(h) \in \varprojlim G_u$ and that it is a homomorphism since all ϕ_u are. □

Example 6. 1. $\varprojlim G$ with all transition maps being the identity map is $\cong G$.

2. $\mathbb{C}[\![X]\!] = \varprojlim \mathbb{C}[X]_n$.

3. For $m \geq n$, consider $\pi_{m,n} : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ sending x to $x \mod p^n$. Then $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is the group of p-adic integers.

4. For $m \mid n$ consider $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ sending x to $x \mod m$. Then $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z}$. [It turns out that the Galois group of every finite field is $\cong \widehat{\mathbb{Z}}$.]

Example 7. An example of the universal property. Consider H the group of power series with coefficients in $\mathbb{C}$ and constant coefficient 1, under multiplication. Let $\phi_n : H \to \mathbb{C}[X]_n$ such that

$$\phi_n(1 + Xf(X)) = \text{truncation of } \sum_{i=1}^{n} \frac{(-1)^{i-1}(Xf(X))^i}{i}$$

This turns out to be a homomorphism. What is ϕ? It is

$$\phi(1 + Xf(X)) = \sum_{i\geq 1} \frac{(-1)^{i-1}(Xf(X))^i}{i} = \log(1 + Xf(X))$$

which is a homomorphism from H to $\mathbb{C}[\![X]\!]$.