Graduate Algebra, Fall 2014
Lecture 21

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1 Group Theory

1.21 Limits (continued)

1.21.2 Inverse limits (continued)

Proposition 1. *Have* $\widehat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p$.

Proof. For p prime and n integer write n_p for the power of p in n. For $x = (x_{p^n}) \in \mathbb{Z}_p$ write

$$x \mod n = \begin{cases} x_{p^{n_p}} & n_p > 0 \\ 1 & n_p = 0 \end{cases}$$

This is clearly a homomorphism $\mathbb{Z}_p \to \mathbb{Z}/p^{n_p}\mathbb{Z}$. The Chinese Remainder Theorem tells us that $\mathbb{Z}/n\mathbb{Z} \cong \prod_p \mathbb{Z}/p^{n_p}\mathbb{Z}$ and we consider $x \mapsto x \mod n$ as the homomorphism $\mathbb{Z}_p \to \mathbb{Z}/p^{n_p}\mathbb{Z} \hookrightarrow \mathbb{Z}/n\mathbb{Z}$.

Consider the homomorphism $\prod \mathbb{Z}_p \to \mathbb{Z}/n\mathbb{Z}$ sending $\prod_p x_p$ to $\prod (x_p \mod n)$. This is a finite product since for $p \nmid n$ we have the convention that $x_p \mod n = 1$. The homomorphism also commutes with the projections $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ for $n \mid m$ and so we get a homomorphism $\prod \mathbb{Z}_p \to \widehat{\mathbb{Z}}$ from the universal property.

Going in the other direction, have a homomorphism $\widehat{\mathbb{Z}} \to \prod \mathbb{Z}_p$ sending (x_n) to $\prod_p (x_{p^k})$. These to homomorphisms are inverses to each other and so they provide an isomorphism. ∎

1.21.3 Duality

Proposition 2. *Suppose* $\{G_u\}$ *is a direct system of groups and* H *is an abelian group. Then*

1. $\{\mathrm{Hom}(G_u, H)\}$ *forms an inverse system of groups and*
2. $\varprojlim \mathrm{Hom}(G_u, H) \cong \mathrm{Hom}(\varinjlim G_u, H)$.

Proof. Define $\pi_{v,u}(f_v) = f_v \circ \iota_{u,v}$ to form the inverse system. An element of $\varprojlim \mathrm{Hom}(G_u, H)$ is a tuple (ϕ_u) for $\phi_u : H \to G_u$ such that $\pi_{v,u} \circ \phi_v = \phi_u$. But by definition this is $\phi_v \circ \iota_{u,v} = \phi_u$ and so by the universal property of direct limits this is equivalent to giving $\phi : \varinjlim G_u \to H$ such that $\phi_u = \phi \circ \iota_u$. Thus we have a bijective homomorphism between $\varprojlim \mathrm{Hom}(G_u, H)$ and $\mathrm{Hom}(\varinjlim G_u, H)$. ∎

1.22 Topological groups

Definition 3. A **topological group** is a group G endowed with a topology wrt which multiplication $m : G \times G \to G$ and inversion $i : G \to G$ are continuous maps. (Here $G \times G$ carries the product topology.)

Example 4.
1. Every group is topological wrt the discrete topology.
2. $S^1 \cong \mathbb{R}/\mathbb{Z}$ is a topological group wrt topology on $\mathbb{C}$.
3. $\mathbb{R}$ with $+$ and the usual topology is a topological group.