Graduate Algebra, Fall 2014
Lecture 22

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1 Group Theory

1.22 Topological groups (continued)

Example 1. 1. Give $\mathbb{R}$ the topology in which only (a, ∞) is open. Then $\mathbb{R}, +$ is not a topological group.

2. If G is a topological group and H is a closed subgroup then G/H is a topological space. If furthermore $H \lhd G$ then G/H is a topological group.

Lemma 2. *If G is a topological group with a dense abelian subgroup then G is abelian.*

Proof. If $x, y \in G$ and $H \subset G$ is the dense abelian subgroup then $x = \lim x_n$ and $y = \lim y_n$ with $x_n, y_n \in H$. Since multiplication and inversion are continuous we find $[x, y] = [\lim x_n, \lim y_n] = \lim[x_n, y_n] = 1$ as H is abelian. Thus G is abelian. □

1.22.1 Profinite groups

Proposition 3. *Suppose $\{G_u\}$ is an inverse system of groups and $G = \varprojlim G_u$. Endow $G \subset \prod G_u$ with the subset topology coming from the product topology on $\prod G_u$.*

1. *Then G is a topological group.*
2. *If G_u is (locally) compact Hausdorff then G is (locally) compact Hausdorff.*
3. *If G_u are all finite then G is said to be* **profinite**. *The group G is compact Hausdorff.*
4. *If $G_u = \langle g_u \rangle$ such that $\pi_{v,u}(g_v) = g_u$ then $\langle (g_u) \rangle \subset G$ is dense, G is said to be* **procyclic** *generated topologically by (g_u).*
5. *If G is profinite then the open subgroups are precisely the subgroups of finite index.*
6. *If G is profinite then all connected components consist of single elements and G is said to be* **totally disconnected**.

Proof. Since the topology on G is the subset topology from the product topology on $\prod G_v$, for an open $U \subset \prod G_v$ we will use the notation U for the open $G \cap U$ throughout this proof.

(1): Suppose $gh = k \in G$ and $k \in U = \prod_{v \in S} U_v \prod_{v \notin S} G_v$ is in an open neighborhood, where S is a finite set (remember that the topology is the product topology). We want to show that $(g, h) \in G \times G$ has an open neighborhood contained in the preimage of U under the multiplication map. Let A_v and B_v be open neighborhoods of g_v respectively h_v such that $A_v \times B_v$ is in the preimage of G_v under the multiplication map in the topological group G_v. Then $A = \prod_{v \in S} A_v \prod_{v \notin S} G_v$ and $B = \prod_{v \in S} B_v \prod_{v \notin S} G_v$ give the open neighborhood $A \times B$ of (g, h) and the image of $A \times B$ under the multiplication map is in U

by construction. Thus the multiplication map is continuous. Also, $U^{-1} = \prod_{v\in S} U_v^{-1} \prod_{v\notin S} G_v$ is open so inversion is continuous as well.

(2): (Local) compactness is a consequence of Tychonoff's theorem since the product topology is then (locally) compact. For Hausdorff, let $g \neq h \in G$ which means $g_v \neq h_v$ for some $v \in I$. Since G_v is Hausdorff can choose U_v and V_v disjoint opens around g_v and h_v. But then $U = U_v \prod_{u\neq v} G_u$ and $V = V_v \prod_{u\neq v} G_u$ are disjoint opens around g and h.

(3): Every finite set is compact under any topology, the discrete topology is Hausdorff.

(4): Skipped.

(5): See homework.

(6): Suppose $S \subset G$ is a connected component containing $g \neq h$. We need a contradiction. Since G is Hausdorff there exists an open neighborhood U around g not containing h. I claim that U contains open open neighborhood gH of g where H is an open subgroup of G. It suffices to check this when $g = 1$ as translating by g is a homeomorphism. By definition U contains an open neighborhood of 1 of the form $\prod_{v\in S} U_v \prod_{v\notin S} G_v$ where S is a finite set. Let $H = \prod_{v\in S}\{1\} \prod_{v\notin S} G_v$, an open subgroup of G (remember, we take $G\cap$). □

Theorem 4. *If G is a compact totally disconnected topological group then G is profinite and $G \cong \varprojlim G/H$ where H ranges over the open subgroups of G, which necessarily have finite index.*