Graduate Algebra, Fall 2014
Lecture 23

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1 Group Theory

1.22 Topological groups (continued)

1.22.1 Profinite groups (continued)

The profinite topology on $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ and $F[\![X]\!] = \varprojlim F[X]_n$ (for any field F) is in fact equivalent to a metric space topology on these sets.

Definition 1. Suppose $I = \mathbb{Z}_{\geq 0}$. For $g \in \varprojlim G_v$ let $\operatorname{ord}(g)$ be the smallest $n \in I$ such that $g_n \neq 0$. If $g_n = 0$ for all n write $\operatorname{ord}(g) = \infty$.

Lemma 2. *For $G = \mathbb{Z}_p$ or $G = F[\![X]\!]$ and $\alpha \in (0, 1)$ define*

$$|g| = \alpha^{\operatorname{ord}(g)}$$

Then $|\cdot|$ is a norm giving a metric space structure on G. Moreover, the metric space topology is equivalent to the profinite topology.

Proof. From the definition $\operatorname{ord}(g+h) \geq \min(\operatorname{ord}(g), \operatorname{ord}(h))$ and so $|g+h| \leq \max(|g|, |h|) \leq |g| + |h|$. Opens are of the form $g \in G$ with $\operatorname{ord}(g) > n$ for some $n \in I$. These are also the opens in the profinite topology, by inspection. □

Example 3.
1. The group $\mathbb{Z}_p$ has as open subgroups the groups $p^n\mathbb{Z}_p$, which generated its topology.
2. The group $\mathbb{F}_p[\![X]\!] = \varprojlim \mathbb{F}_p[X]/(X^n)$ as as open subgroups $X^n\mathbb{F}_p[\![X]\!]$, which generate the topology.

1.22.2 Pontryagin duals

Definition 4. For a topological abelian group G let $\widehat{G}$ be the abelian group of all continuous homomorphisms $G \to S^1$. This is typically a smaller set that the dual G^* as the latter contains all homomorphisms, not only the continuous ones. The abelian group $\widehat{G}$ is called the Pontryagin dual of G and is the natural setting for Fourier analysis on abelian groups.

Example 5.
1. $\widehat{\mathbb{R}} \cong \mathbb{R}$ sending $x \in \mathbb{R}$ to the continuous homomorphism $e_x(y) = e^{2\pi i x y}$.
2. $\widehat{\mathbb{R}/\mathbb{Z}} \cong \mathbb{Z}$ sending $n \in \mathbb{Z}$ to $e_n(x) = e^{2\pi i x n}$.

The power of Pontryagin duals is that they too can be made into topological groups.

Proposition 6. *Let G be a topological abelian group and $\widehat{G}$ its Pontryagin dual. For a compact subset $K \subset G$ and an open $U \subset S^1$ (here S^1 is endowed with the subset topology from $S^1 \subset \mathbb{C}$) let $W(K, U) = \{f \in \widehat{G} | f(K) \subset U\}$. Consider the smallest topology on $\widehat{G}$ in which all $f \cdot W(K, U)$ are open sets as $f \in \widehat{G}$, K is compact and U is open vary. Then $\widehat{G}$ is a locally compact topological abelian group.*

Proof. We first need to check that in the topology defined on $\widehat{G}$ the multiplication and inversion maps are continuous. Since the topology on $\widehat{G}$ is generated by $W(K,U)$ inversion is continuous if $W(K,U)^{-1}$ is open. But $W(K,U)^{-1} = W(K,U^{-1})$ and $U^{-1} \subset S^1$ is open because S^1 is a topological group.

Continuity of multiplication: next time. $\square$