Graduate Algebra, Fall 2014
Lecture 24

Andrei Jorza

2014-10-27

1 Group Theory

1.22 Topological groups (continued)

1.22.2 Pontryagin duals (continued)

Proposition 1. *Let G be a topological abelian group and $\widehat{G}$ its Pontryagin dual. For a compact subset $K \subset G$ and an open $U \subset S^1$ (here S^1 is endowed with the subset topology from $S^1 \subset \mathbb{C}$) let $W(K, U) = \{f \in \widehat{G} | f(K) \subset U\}$. Consider the smallest topology on $\widehat{G}$ in which all $f \cdot W(K, U)$ are open sets as $f \in \widehat{G}$, K is compact and U is open vary. Then $\widehat{G}$ is a locally compact topological abelian group.*

Proof. Continuity of inversion: last time.

Let's check that multiplication is continuous. Suppose $fg = h \in \eta W(K, U)$, for some $\eta \in \widehat{G}$. We would like to show that there exists an open neighborhood of (f, g) in $\widehat{G} \times \widehat{G}$ contained inside the set $\{(\phi, \gamma) \in \widehat{G} \times \widehat{G} | \phi\gamma \in \eta W(K, U)\}$.

Since S^1 is a topological group, the preimage of U under multiplication is open and, in particular, it contains an open set $V_1 \times V_2$, with $V_1 V_2 \subset U$. Then $fW(K, V_1) \times gW(K, V_2)$ via multiplication is $hW(K, V_1)W(K, V_2) \subset hW(K, U) \subset \eta W(K, U)$ as desired. ☐

Theorem 2 (Pontryagin duality)**.** *If G is an abelian topological group then $\widehat{\widehat{G}} \cong G$.*

This is not easy.

2 Rings

2.1 Basics

2.1.1 Definitions

Definition 3. A **ring** (with unit) is a set R together with binary operations $+$ and $\cdot$ such that $(R, +)$ is an abelian group and $(R, \cdot)$ is a monoid and multiplication is distributive wrt addition.

Example 4.

1. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are rings, but $\mathbb{N}$ is not.
2. If $n \geq 2$ then $\mathbb{Z}/n\mathbb{Z}$ is a ring wrt addition and multiplication mod n.
3. If R is a ring then $R[X]$ is also a ring.
4. If X is a set and R is a ring then the set of functions $\{f : X \to R\}$ is a ring. (E.g., if X is a topological space then continuous/smooth/differentiable/integrable functions $f : X \to \mathbb{R}$ forms a ring as well.)
5. If R is a ring then $R[\![X]\!]$ is a ring.

6. If R is a ring then $n \times n$ matrices with entries in R form a noncommutative ring $M_n(R)$.

Definition 5. Let R be a ring.

1. R is a **division ring** if $(R - 0, \cdot)$ is a group.
2. R is **commutative** if $(R, \cdot)$ is commutative.
3. A commutative division ring is a **field**.

Example 6. 1. The real hamiltonians $\mathbb{H} = \{a + bi + cj + dk | a, b, c, d \in \mathbb{R}\}$, addition component-wise and multiplication given by $i^2 = j^2 = k^2 = -1$ and $ij = k$. Then if $(a, b, c, d) \neq 0$ we get the inverse

$$(a + bi + cj + dk)^{-1} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2}$$

so $\mathbb{H}$ is a division ring.

2. Let G be a group and R a ring. Then $R[G] = \{\sum a_g[g]\}$ with component-wise addition and multiplication given by $\sum a_g[g] \sum b_h[h] = \sum a_g b_h[gh]$ is a ring with unit $[1]$. It is commutative if and only if both R and G are.

   For example $R[\mathbb{Z}] \cong R[X, X^{-1}]$ the ring of Laurent polynomials via the map $\sum a_n[n] \mapsto \sum a_n X^n$.

3. $\mathbb{Z}$ is not a field.

4. $M_n(R)$ is not a division ring since the matrix with 1 in the top right corner and 0 elsewhere is a 0 divisor (it has square 0).

2.1.2 Integral Domains

Definition 7. Let R be a ring.

1. $x \in R$ is a **zero divisor** if $xy = 0$ or $yx = 0$ for some $y \neq 0$.
2. R is a **(integral) domain** if it has no zero divisors.

Example 8. 1. $\mathbb{Z}$ is a domain.

2. $\mathbb{Z}/p\mathbb{Z}$ is a domain for p prime.
3. $\mathbb{Z}/n\mathbb{Z}$ is not a domain if n is not a prime.
4. $M_n(R)$ is not a domain.
5. If R is a domain then $R[X]$ and $R[\![X]\!]$ are integral domains. Indeed, if $f(X) = aX^n + O(X^{n+1})$ and $g(X) = bX^m + O(X^{m+1})$ then $f(X)g(X) = abX^{m+n} + O(X^{m+n+1})$ and if this is 0 then $ab = 0$ which implies that a or b is 0.