Graduate Algebra, Fall 2014
Lecture 27

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2 Rings

2.3 Special types of ideals (continued)

2.3.2 Radicals (continued)

Proposition 1. *If $I \subset R$ is an ideal then $\sqrt{I}$ is an ideal of R. In particular* $\mathrm{Nil}(R)$ *is an ideal of R.*

Proof. A subset $J \subset R$ is an ideal iff for $x, y \in J$ and $r \in R$ have $x+yr \in J$. Suppose $x, y \in \sqrt{I}$ so $x^n, y^m \in I$ for $n, m \geq 1$ and let $r \in R$. Then

$$(x+yr)^{n+m} = x^n \sum_{k=0}^{m} \binom{m+n}{k} x^{m-k} r^k y^k + y^m \sum_{k=m+1}^{n+m} \binom{m+n}{k} x^{n+m-k} r^k y^{k-m} \in I$$

□

Proposition 2. *Let R be a commutative ring and I an ideal. Then*

1. $\mathrm{Nil}(R) = \cap \mathfrak{p}$ *is the intersection of all prime ideals of R.*
2. $\sqrt{I}$ *is the intersection of all prime ideals of R containing I.*

Proof. (1): If $x \in \mathrm{Nil}(R)$ then $x^n = 0 \in \mathfrak{p}$ for any prime ideal $\mathfrak{p}$. Thus $x \in \mathfrak{p}$ and we deduce that $\mathrm{Nil}(R) \subset \cap \mathfrak{p}$. Reciprocally, suppose $x \in \cap \mathfrak{p}$ but x is not nilpotent.

Let S be the set of ideals $I \subsetneq R$ not containing any positive power of x or, equivalently, ideals I such that $x^n \notin I$ for $n \geq 0$. Since $x^n \neq 0$ for all n, at least S contains the trivial ideal (0) so S is not empty. Again, if T is a totally ordered subset of S, then $I_T = \cup_{I \in T} I$ is an ideal $I_T \subsetneq R$ (see last lecture). If $x^n \in I_T$ then $x^n \in I \in T$ for some I, which is not the case as $I \in S$, and so $I_T \in S$, not containing any power of x. Again by Zorn's lemma we deduce that S has a maximal element $\mathfrak{p} \subsetneq R$.

It's enough to show that $\mathfrak{p}$ is a prime ideal. Suppose $a, b \notin \mathfrak{p}$ but $ab \in \mathfrak{p}$. Then $\mathfrak{p} + (a)$ and $\mathfrak{p} + (b)$ are bigger than $\mathfrak{p}$ and so they contain powers of x. Let $x^m \in \mathfrak{p} + (a)$ and $x^n \in \mathfrak{p} + (b)$ for $m, n \geq 0$. Then $x^{m+n} \in \mathfrak{p} + (ab) = \mathfrak{p}$ giving a contradiction as $\mathfrak{p} \in S$. This implies that $x \notin \mathfrak{p}$ for the prime ideal $\mathfrak{p}$, contradicting the choice of x.

(2): Consider $R \to R/I$. The image of $\sqrt{I}$ is, by definition, $\mathrm{Nil}(R/I)$ and so is the intersection of the prime ideals of R/I. From the homework, this is the intersection of the prime ideals of R containing I as desired. □

Lemma 3. *A* **unit** *in a ring R is an element $x \in R$ which has an inverse $y \in R$, i.e., such that $xy = 1$.*

1. *The set $R^\times$ of units is a group.*
2. *If $u \in R^\times$ and I is an ideal such that $u \in I$ then $I = (1) = R$.*

3. *If* $u \in R^\times$ *and* I *is an ideal then* $(u)I = I$.

Proof. If $xy = 1$ and $uv = 1$ then $xu(vy) = 1$ so $xu \in R^\times$ and if $x \in R^\times$ with inverse y then $y \in R^\times$ with inverse x so $R^\times$ is a group.

If $u \in R^\times \cap I$ then for all $r \in R$, $ru^{-1} \in R$ so $r = ru^{-1}u \in RI = I$ so $R = I$. This proves the last two parts. □

Example 4. 1. $\mathbb{Z}^\times = \{\pm 1\}$.

2. If F is a field then $F^\times = F - 0$.

3. $\mathbb{C}[X]^\times = \mathbb{C} - 0$ as a polynomial has an inverse which is also a polynomial only if it has degree 0.

4. From the homework, $R[\![X]\!]^\times$ consists of power series $a_0 + a_1X + \cdots$ such that $a_0 \in R^\times$.

Example 5. Let $R = \mathbb{C}[X]/(X^2) = \{a + bX | a, b \in \mathbb{C}\}$. If $a \neq 0$ then $(a + bX)(a^{-1} - ba^{-2}X) = 1$ in R so $a + bX \in R^\times$.

If I is an ideal of R containing some $a+bX$ then the lemma says that $I = R$. If I only contains expressions of the form bX then either $I = (0)$ or $I = (bX) = (X)$ if $b \neq 0$. So the ideals of R are either 0, (X) or R. Thus the unique prime ideal is (X).

What about $\mathrm{Nil}(R)$? The theorem says it should be (X). Note that $(a + bX)^n = a^n + na^{n-1}bX$ which is 0 iff $a = 0$ iff $a + bX \in (X)$ so $\mathrm{Nil}(R) = (X)$ from the definition.

Example 6. What is $\sqrt{(x, y^3)}$ in $\mathbb{C}[x, y]$? From last time this is (x, y). If $\mathfrak{p} \subset \mathbb{C}[x, y]$ is a prime ideal containing x, y^3 then it contains x, y, from primality, so $\mathfrak{p}$ contains (x, y) which is a maximal ideal.

What about $\sqrt{(y^3)}$? Certainly (y) contains (y^3) and is a prime ideal. Thus $\sqrt{(y^3)} \subset (y)$. But also $(yP(x, y))^3 \in I$ so $(y) \subset \sqrt{(y^3)}$ so we have $\sqrt{(y^3)} = (y)$.

Remark 1. Algebraic geometry starts with Hilbert's Nullstellensatz which states that if $I = (P_1(X_j), \ldots, P_m(X_j))$ is an ideal of $\mathbb{C}[X_1, \ldots, X_n]$ then $\sqrt{I}$ will be generated by irreducible polynomials $Q_j(X_1, \ldots, X_n)$ such that the set of solutions in $\mathbb{C}^n$ of the systems of equations $P_i(X_j) = 0$ and $Q_i(X_j) = 0$ are the same.

For example, $\sqrt{(x, y^3)} = (x, y)$ because we seek irreducible polynomials whose set of common roots are the solutions to $x = 0$ and $y^3 = 0$, i.e., $x = y = 0$, and the polynomials x and y work.