Graduate Algebra, Fall 2014
Lecture 28

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Started with a panoramic view of what we'll do in ring theory.

2 Rings

2.3 Special types of ideals (continued)

2.3.2 Radicals (continued)

Lemma 1. *If $\mathfrak{m}$ is a maximal ideal of R then $\mathfrak{m}$ is also maximal wrt inclusion, i.e., if I is such that $\mathfrak{m} \subset I \subset R$ then I is either $\mathfrak{m}$ or R.*

Proof. $I/\mathfrak{m}$ is an ideal of $R/\mathfrak{m}$ which is a field. Thus $I/\mathfrak{m}$ is either 0 (in which case $I = \mathfrak{m}$) or all of $R/\mathfrak{m}$ in which case $R/I \cong (R/\mathfrak{m})/(I/\mathfrak{m}) = 0$ so $I = R$. □

Definition 2. The **Jacobson** radical of R is $J(R) = \cap \mathfrak{m}$ the intersection of all maximal ideals of R.

Proposition 3. *Let R be a commutative ring.*

1. $\mathrm{Nil}(R) \subset J(R)$.
2. *$x \in J(R)$ if and only if $1 - xy \in R^\times$ for all y.*

Proof. (1): trivial.

(2): If $1 - xy$ is not a unit then $(1 - xy) \subset \mathfrak{m}$ for some $\mathfrak{m}$ maximal and so, since $x \in \mathfrak{m}$, we get $1 \in \mathfrak{m}$, a contradiction. Reciprocally, if $x \notin \mathfrak{m}$ for some maximal $\mathfrak{m}$, then $\mathfrak{m} \subsetneq \mathfrak{m} + (x)$ and by maximality get $\mathfrak{m} + (x) = R$. Thus $1 \in \mathfrak{m} + (x)$ so $1 = a + xy$ for some $a \in \mathfrak{m}$ which implies that $a = 1 - xy \in \mathfrak{m}$ is not a unit. □

2.4 Pullbacks and pushforwards of ideals

For a ring R denote by $\mathcal{P}_R \subset \mathcal{I}_R$ be the set of prime resp all ideals of R.

Proposition 4. *Let $f : R \to S$ be a ring homomorphism.*

1. *The map f^* defined as $f^*(J) = f^{-1}(J)$ gives a map $f^* : \mathcal{I}_S \to \mathcal{I}_R$ which restricts to $f^* : \mathcal{P}_S \to \mathcal{P}_R$.*
2. *The map f_* defined as $f_*(I) = f(I)S$ (defined as the ideal generated by $\{f(x)s | x \in I, s \in S\}$) gives a map $f_* : \mathcal{I}_R \to \mathcal{I}_S$. However, f_* need not take prime ideals to prime ideals.*

Proof. (1): Consider the composite $\overline{f} : R \to S \to S/J$. Then $\ker \overline{f} = f^*(J)$ and so $f^*(J)$ is an ideal. Moreover, $R/f^*(J) \cong \mathrm{Im}(\overline{f}) \subset S/J$. If J is prime then S/J is integral domain and so $R/f^*(J) \cong \mathrm{Im}\, f \subset S/J$ is also an integral domain. However, if J were maximal we'd get $R/f^*(J)$ is a subring of a field which need not be a field.

(2): This is tautological. □

Example 5. 1. Consider $f : \mathbb{Z} \hookrightarrow \mathbb{C}$ and $J = (0)$ which is maximal in $\mathbb{C}$ but $f^*((0)) = (0)$ is not maximal in $\mathbb{Z}$.

2. Consider the injection $f : \mathbb{Z} \hookrightarrow \mathbb{Z}[i]$ and $\mathfrak{p} = (2)$. Then $f_*(I) = (2)\mathbb{Z}[i]$ is not a prime ideal since $2 = -i(1+i)^2$ and so $(1+i)^2 \in f_*(I)$ but $1 + i \notin f_*(I)$ since if $1 + i = 2x$ then $x = (1+i)/2 \notin \mathbb{Z}[i]$.