Graduate Algebra, Fall 2014
Lecture 29

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2 Rings

2.5 Rings of fractions and localization

Proposition 1. *Let R be a commutative ring and $S \subset R$ be a subset closed under multiplication and such that $1 \in S$. Consider the relation $(a, s) \sim (b, r)$ on $R \times S$ iff $(ar - bs)u = 0$ for some $u \in S$.*

1. *The equivalence relation $\sim$ is reflexive and transitive.*
2. *Write a/s for the equivalence class (a, s). Define $S^{-1}R = \{a/s | a \in R, s \in S\}$. The operations $a/s + b/r = (ar + bs)/(rs)$ and $(a/s)(b/r) = (ab)/(sr)$ are well-defined and turn $S^{-1}R$ into a ring.*
3. *The natural map $f : R \to S^{-1}R$ sending x to $x/1$ is a ring homomorphism.*

Example 2. If R is an integral domain and $S = R - 0$ then $S^{-1}R$ is a field, called the fraction field $\operatorname{Frac} R$.

Proposition 3. *Suppose $g : R \to T$ be a ring homomorphism such that for $u \in S$, $g(u) \in T^\times$ is a unit. Then g factors through $S^{-1}R$, i.e., there exists a unique $h : S^{-1}R \to T$ such that $g = h \circ f$.*

Proof. Given h, note that $h(a/s) = h(a/1)h(1/s) = g(a)h(s)^{-1} = g(a)g(s)^{-1}$ so h is uniquely determined.

Now define $h(a/s) = g(a)g(s)^{-1}$. It's not hard to check that h is well-defined and a ring homomorphism. □

Example 4.

1. $S^{-1}R = 0$ iff $0 \in S$.
2. If $x \in R$ then $S = \{x^n | n \geq 0\}$ is multiplicatively closed and $R_f = S^{-1}R$. In the case of $R = \mathbb{Z}$, this is what we called $\mathbb{Z}[1/f]$.
3. If I is an ideal of R then $S = 1 + I$ is multiplicatively closed.
4. If $\mathfrak{p}$ is a prime ideal then $S = R - \mathfrak{p}$ is multiplicatively closed in R. The **localization** of R at $\mathfrak{p}$ is $R_\mathfrak{p} = S^{-1}R$.
5. $\mathbb{Z}_{(p)}$ consists of rationals whose denominators are not disible by p.
6. For $R = \mathbb{C}[x, y]$ and $\mathfrak{p} = (x, y)$ the localization $R_\mathfrak{p}$ consists of rational functions $P(x, y)/Q(x, y)$ such that $Q \notin (x, y)$, i.e., such that $Q(0, 0) \neq 0$.

Proposition 5. *If R is an integral domain defined* $\operatorname{Frac} R = S^{-1}R$ *where $S = R - 0$. Then* $\operatorname{Frac} R$ *is a field and $R \to$* $\operatorname{Frac} R$ *is injective.*

Proof. Suppose x/r is nonzero in $\operatorname{Frac} R$. Then $x \neq 0$ and so $s/x \in \operatorname{Frac} R$ is an inverse. Suppose $x/1 = 0$ in $\operatorname{Frac} R$. Then $xs = 0$ for some $s \neq 0$ which, since R is an integral domain, implies $x = 0$ and so $R \hookrightarrow \operatorname{Frac} R$. □

Proposition 6. *Let R be a commutative ring and S a multiplicatively closed subset containing 1. Consider the map $f : R \to S^{-1}R$ sending $x \mapsto x/1$.*

1. *If $I \subset R$ is an ideal then $f_*(I) = S^{-1}I$ and $f_* : \mathcal{I}_R \to \mathcal{I}_{S^{-1}R}$ is surjective.*
2. *If $\overline{S}$ is the image of S in R/I then $S^{-1}R/S^{-1}I \cong \overline{S}^{-1}(R/I)$.*
3. *There is a bijection $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ between the prime ideals of R which are disjoint from S and the prime ideals of $S^{-1}R$.*
4. *If $\mathfrak{p}$ is a prime ideal of R then there is a bijection between the prime ideals of $R_\mathfrak{p}$ and the prime ideals of R contained in $\mathfrak{p}$. This is a tool for constructing rings with few prime ideals.*

Proof. (1): If $a \in I$ and $b/r \in S^{-1}R$ then $(a/1)(b/r) = (ab)/r \in S^{-1}I$ and so $S^{-1}I \subset f_*(I)$. Moreover, it is easy to check that $S^{-1}I$ is an ideal and so $f_*(I) = S^{-1}I$. Suppose that $J \subset S^{-1}R$ and let $I = \{a \in R | a/1 \in J\}$. If $a/s \in J$ then $a/1 = (s/1)(a/s) \in J$ so $a \in I$ and if $a \in I$ then $a/1 \in J$ and so $(1/s)(a/1) = a/s \in J$. Thus $J = S^{-1}I$.

(2): The map $a/s \mapsto (a+I)/(s+I)$ is a well defined homomorphism from $S^{-1}R$ to $\overline{S}^{-1}R/I$. It is clearly surjective. The kernel contains a/s iff $(a+I)/(s+I) = 0$ iff $(a+I)(r+I) = 0$ for some $r \in S$. But then $ar \in I$ and so $a \in S^{-1}I$ giving $a/s \in S^{-1}I$. The result then follows from the first isomorphism theorem.

(3): We need that $\mathfrak{p} \mapsto f_*(\mathfrak{p})$ is a bijection from prime ideals $\mathfrak{p} \cap S = \emptyset$ and prime ideals of $S^{-1}R$. For a prime ideal $\mathfrak{p}$ of R we know that $S^{-1}R/f_*(\mathfrak{p}) \cong \overline{S}^{-1}(R/\mathfrak{p})$. Since $R/\mathfrak{p}$ is an integral domain, it follows that $S^{-1}R/f_*(\mathfrak{p}) \subset \operatorname{Frac} R/\mathfrak{p}$ and so $f_*(\mathfrak{p})$ is either prime ideal or is all of $S^{-1}R$. The latter can happen iff $\overline{S}^{-1}R/\mathfrak{p} = 0$ iff $0 \in \overline{S}$ iff $\mathfrak{p} \cap S \neq \emptyset$.

This shows that f_* takes prime ideals disjoint from S to prime ideals of $S^{-1}R$.

Going in the other direction, suppose $\mathfrak{q}$ is a prime ideal of $S^{-1}R$ and let $\mathfrak{p} = f^*(\mathfrak{q})$ the corresponding prime ideal of R. If $x \in \mathfrak{p} \cap S$ then $1 = (x/1)(1/x) \in \mathfrak{q}S^{-1}R = \mathfrak{q}$ so $\mathfrak{q} = S^{-1}R$ contradicting that $\mathfrak{q}$ is a prime ideal. Thus $\mathfrak{p} \cap S = \emptyset$. What is $f_*(\mathfrak{p})$? Certainly $S^{-1}\mathfrak{p} \subset \mathfrak{q}$ as $\mathfrak{q}$ is an ideal. If $a/s \in \mathfrak{q}$ then $a/1 \in \mathfrak{q}$ so $a \in \mathfrak{p}$ and so $a/s \in S^{-1}\mathfrak{p}$ which gives $\mathfrak{q} = S^{-1}\mathfrak{p} = f_*(\mathfrak{p})$.

This shows that $\mathfrak{p} \mapsto f_*(\mathfrak{p})$ is a bijection as desired.

(4): Apply (3) to $S = R - \mathfrak{p}$. ☐

Example 7. Suppose $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \subset R$ are prime ideals. Let $S = R - \cup\mathfrak{p}_i$ and set $R_{\mathfrak{p}_1,\ldots,\mathfrak{p}_n} = S^{-1}R$. Then the prime ideals of $R_{\mathfrak{p}_i}$ are in bijection with the prime ideals of R contained in $\cup\mathfrak{p}_i$.

Example 8. Let n be square-free. Then $\mathbb{Z}_{(n)} = \{\frac{a}{b} | (b,n) = 1\}$ has prime ideals (p) for $p \mid n$ only.