Graduate Algebra, Fall 2014
Lecture 33

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3 Modules

Complexes of modules and exact sequences

Definition 1. A **complex** of R-modules is a collection of R-modules (M_i) indexes by $i \in \mathbb{Z}$ together with R-module homomorphisms $f_i : M_i \to M_{i+1}$ such that $f_{i+1} \circ f_i = 0$.

Example 2.
1. Let $f : M \to N$ be an R-module homomorphism and let $\pi : N \to N/M$ be the projection map. Then $M \xrightarrow{f} N \xrightarrow{\pi} N/M$ is a complex.
2. More generally, if $f : M \to N$ and $g : N \to L$ such that $g \circ f = 0$ then $0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ is a complex.

Definition 3. A complex $\ldots \to M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \to \ldots$ is **exact** at M_i if $\ker f_i = \operatorname{Im} f_{i-1}$. A complex is said to be an **exact sequence** if it is exact at every module in the sequence.

Example 4.
1. The complex $M \to N \to N/M$ is exact at N since $\ker(N \to N/M) = M = \operatorname{Im}(M \to N)$.
2. $0 \to M \xrightarrow{f} N$ is exact at M iff f is injective.
3. $M \xrightarrow{f} N \to 0$ is exact at N iff f is surjective.

Definition 5. A short exact sequence is an exact sequence of the form $0 \to M \to N \to L \to 0$. The first isomorphism theorem can be rephrased as $L \cong N/M$.

3.3 Noetherian rings and modules (continued)

Definition 6. A ring R is **Noetherian** if every ideal is Noetherian.

Example 7.
1. PIDs are Noetherian.
2. $\mathbb{Z}[X_1, X_2, \ldots]$ is not Noetherian since the ideal $(X_1, \ldots)$ is not finitely generated.

Proposition 8. *An R module M is Noetherian iff every submodule of M is finitely generated.*

Proof. Suppose every submodule is finitely generated. Let $M_1 \subset \ldots$ be an ascending chain and $N = \cup M_i$. Since N is finitely generated, its generators are in some big M_n and so the chain is stationary.

Let N be a submodule of M and S the set of finitely generated submodules of N. Then S is not empty since $0 \in S$. Also, every ascending chain in S has a max since the chain is stationary. Thus by Zorn's lemma S has a maximal finitely generated submodule N' of N. If $N \neq N'$ choose $n \in N - N'$ in which case $N' \subsetneq N' + nR \subset N$ is a larger finitely generated submodule, contradicting the choice of N'. Thus $N = N'$ is also finitely generated. $\square$

Definition 9. A short exact sequence of R-modules is $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ such that f and g are R-module homomorphisms, f is injective, g is surjective and $\operatorname{Im} f = \ker g$.

Example 10. If $f : M \to N$ is injective then $0 \to M \to N \to N/f(M) \to 0$ is short exact.

Proposition 11. *Let $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ be an exact sequence of R-modules. Then N is Noetherian (resp. Artinian) iff both M and P are.*

Proof. Only the Noetherian part: Suppose that N is Noetherian and M_i is an ascending chain in M. Then $f(M_i)$ is an ascending chain in N and thus is stationary. Since f is injective, M_i has to be stationary. If P_i is an ascending chain in P then $g^{-1}(P_i)$ is an ascending chain in N and thus is stationary. By surjectivity of g, it follows that $g(g^{-1}(P_i)) = P_i$ and so P_i is stationary.

Now suppose that M and P are Noetherian. Let N_i be an ascending chain. Then $f^{-1}(N_i)$ is stationary, say with limit A, and $g(N_i)$ is stationary, say with limit B. But $0 \to f^{-1}(N_i) \to N_i \to g(N_i) \to 0$ is exact so for i large $0 \to A \to N_i \to B \to 0$ is exact. Then N_i is stationary by the third isomorphism theorem. (Indeed, $N_{i+1}/N_i \cong (N_{i+1}/A)/(N_i/A) \cong B/B = 0$.) □

Proposition 12. *If R is a Noetherian ring the M is Noetherian iff it is finitely generated.*

Proof. If Noetherian then every submodule, and so M itself, is finitely generated.

Suppose finitely generated. Then there exists a surjection $R^n \to M$.

The free module R is Noetherian because its submodules are ideals and R is a Noetherian ring. By induction $0 \to R \to R^n \to R^{n-1} \to 0$ so R^n is Noetherian. The previous proposition then gives M is Noetherian. □